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A COMPARISON OF SEQUENTIAL TESTS FOR THE POISSON PARAMETER

by

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INTRODUCTION

This investigation treats a truncated sequential test for testing a simple hypothesis against a simple alternative. Attention is confined to the case of sampling from a Poisson population. Although Wald's [11][1] sequential probability ratio test (SPRT) will terminate with probability one at some stage of an experiment, there is no guaranteed upper bound for the sample size of this test. However, for cases in which time and cost of sampling are involved we often have to face the situation of making a definite decision within a given number of trials. With this in mind, several sequential tests, such as Anderson's [1], Armitage's [2] and Donnelly's [3], have been proposed which are truncated within some fixed stage. However, none of these are directly applicable to Poisson sampling.

Here, as a variation of the SPRT, we treat Hall's [4] minimum probability ratio test (MPRT). Whereas for the SPRT the rejection and acceptance lines are parallel, for the MPRT these two lines converge so that the test always terminates with a definite decision by a predetermined stage.

More specifically, we present and compare several test procedures for testing whether a Poisson parameter has value λ_1 or λ_2 (specified numbers) based on a sequence of independent observations from a Poisson population. The test procedures considered here are:

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[1]The numbers in square brackets refer to the bibliography listed at the end.

i) the sequential probability ratio test, both untruncated (SPRT) and truncated ($SPRT_o$)(Chapter I),

ii) the minimum probability ratio test (MPRT) (Chapter II), and

iii) the most powerful fixed sample size test (FSST) (Chapter III).

A diagram for carrying out these tests appears in Figure I[2](A.2).

A requirement in all of these test procedures is that each error probability should not exceed a common specified level α. The bases for comparison of the test procedures are the operating characteristic (OC) function, the expected sample size (ASN) function, and the standard deviation of the sample size (SDN) function. Hoeffding's lower bound [5] (HLB) on the ASN at an intermediate value is also presented. (Chapter III)

Major attention is given to the MPRT since the other procedures are quite well known. An extensive discussion of this test procedure appears in Chapter II. Actually, the MPRT can only guarantee an upper bound on an average of the two error probabilities. Achieving equal error probabilities hinges on the choice of an intermediate λ_o value. λ_o is here chosen to make equal the divergence (a concept from information theory) between λ_1 and λ_o and between λ_o and λ_2 . This value of λ_o is denoted D. Some consideration is also given to $\lambda_o = s$, the slope of the SPRT acceptance lines. The calculations indicate that either of these choices is quite successful, D being slightly better for moderate or large α values and s being slightly better for small α values ($\leq .01$). (See the tables and Chapter V.)

[2]All figures and tables appear in the Appendix.

For the MPRT, calculation of seven points on the OC curve, ASN curve and SDN curve were carried out, $\alpha = .1, .05, .01$ and $.001$, for the following pairs of hypotheses:

Table	λ_1	λ_2	λ_0
I	.1	.3	D
II	.5	.8	D
III	.5	1.	D
IV	5.	8.	D
V	.1	.5	D and s
VI	1.	2.	D and s

Also presented in Table I and VI are the sample size n_F for the FSST, the maximum sample size n_o of the MPRT, Hoeffding's lower bound at D and s, and the ASN of the SPRT at four values, 0, λ_1, s, and λ_2, calculated by Wald's approximation [11]. The exact calculations were performed on the UNIVAC 1105[3] by a program described in Chapter IV and A.7.

For purposes of comparing the three test procedures, the calculation of 16 points on the OC curve, ASN curve, and SDN curve were carried out for all tests, for $\alpha = .1, .05, .01$, and $.001$, and for the following pairs of hypotheses:

Table	Figure	λ_1	λ_2
VII	2	.01	.1
VIII	3	2.	4.

[3]In this thesis the computer refers to UNIVAC 1105.

Exact calculations were done for the OC, ASN and SDN functions of the MPRT ($\lambda_o = D$) and the SPRT truncated at n_o, the maximum sample size of the MPRT, and Hoeffding's lower bound at λ = s and D by the computer. Also calculated were three values of Wald's approximations to the OC and ASN functions for the untruncated SPRT. The FSST calculation are based on Poisson tables [8] to the extent available and otherwise on the normal approximation. A discussion of the results of the calculations appears in Chapter V.

One use for tests concerning a Poisson population is in quality control work where the defects in a unit are counted and the quality of a lot or process is judged by the average number of defects per unit. This differs from the test where each unit is placed into a "defective" or "non-defective" category and the quality of a lot or process is determined by the total number of defectives.

The defects per unit analysis is useful under the following conditions.

i) If almost every unit contains at least one defect, a fraction defective plan ... "defective" or "non-defective" classification ... obviously not feasible for such a case.

ii) For products which are expensive to produce or inspect and products customarily inspected in small lots, it is too costly to obtain samples large enough to assure high discrimination by a fraction defective plan. However, if the number of samples observed is sufficiently large and the quality of the product is high, the result will not differ greatly between these two tests.

iii) For this test to apply exactly, the defects must be randomly and independently distributed.

CHAPTER I

SEQUENTIAL PROBABILITY RATIO TEST

AND ITS APPLICATION TO THE POISSON DISTRIBUTION

1.1. Wald's Sequential Probability Ratio Test (SPRT) for Testing a Simple Hypothesis against a Simple Alternative [11].

Suppose x_1, x_2, ... is a sequence of independent observations with a common density function $f(x:\theta)$ and let $X_m = (x_1, x_2, \dots x_m)$, $m = 1, 2, \dots$, be a sample of size m. Let α_1 and α_2 be, respectively, the desired probability of accepting the alternative hypothesis $H_2:\theta = \theta_2$ when the true parameter is θ_1, and of accepting the null hypothesis $H_1:\theta = \theta_1$ when the true parameter is θ_2, and call (α_1, α_2) the strength of the test. We also denote by d_i the decision to accept $H_i (i = 1. 2)$.

For any positive integer m, the joint density function f_{im} of a sample of size m under H_i $(i = 1. 2)$ is

$$f_{im} = f(x_1:\theta_i) \cdot f(x_2:\theta_i) \dots f(x_m:\theta_i) \tag{1.1.1}$$

Then, the SPRT is carried out as follows:

For suitably chosen A and B $(0 < B < 1 < A)$, at the m-th stage $(m = 1. 2 \dots)$ of the experiment,

(i) stop sampling and accept H_1 if

$$B < f_{2j}/f_{1j} < A \quad (j = 1.2..m-1) \text{ and } f_{2m}/f_{1m} \leq B \tag{1.1.2}$$

(ii) stop sampling and accept H_2 if

(1.1.3) $\qquad B < f_{2j}/f_{1j} < A \ (j=1.2\ldots m-1)$ and $f_{2m}/f_{1m} \geq A$

(iii) otherwise, continue sampling until f_{2m}/f_{1m} falls into either category (i) or (ii).

The calculation of A and B to obtain the desired strength (α_1, α_2) is very laborious. Therefore, in practice Wald suggested putting

(1.1.4) $\qquad A' \equiv (1 - \alpha_2)/\alpha_1$ and $B' \equiv \alpha_2/(1 - \alpha_1)$

as substitutes for A and B, respectively.

Denoting the resulting error probabilities by α_1' and α_2' we can easily see that

(1.1.5) $\qquad \alpha_1' + \alpha_2' \leq \alpha_1 + \alpha_2$

and therefore, at least one of the inequalities, $\alpha_1' \leq \alpha_1$ and $\alpha_2' \leq \alpha_2$, must be satisfied. Moreover, if $\alpha_1 = \alpha_2 = \alpha$, both inequalities are almost achieved.

The most important characteristics of the SPRT are the operating characteristic (OC) and the Average Sample Number (ASN) functions. The OC function, $L(\theta)$, is defined as the probability of accepting the null hypothesis H_1 when the true parameter is θ. An approximation [11] is given by

(1.1.6) $$L(\theta) \sim \frac{A^{h(\theta)} - 1}{A^{h(\theta)} - B^{h(\theta)}}$$

where h satisfies

(1.1.7) $$Ee^{hz} = 1$$

with $z = \log f(x:\theta_2)/f(x:\theta_1)$. If $h = 0$ is the only solution of (1.1.7), the right hand side of (1.1.6) is evaluated by taking the limit using l'Hospital's Rule.

The ASN function, $E_\theta(n)$, is the expectation of the sample size when θ is the true parameter and is approximately given [11] by

$$(1.1.8) \qquad E_\theta(n) \sim \left\{L(\theta)\log B + [1-L(\theta)]\log A\right\} / E_\theta(z) \text{ if } E_\theta(z) \neq 0$$

and

$$(1.1.9) \qquad E_\theta(n) \sim \left\{L(\theta)(\log B)^2 + [1-L(\theta)](\log A)^2\right\} / E_\theta(z) \text{ if } E_\theta(z) = 0$$

1.2. <u>The SPRT for the Poisson Distribution</u>.

Suppose x_1, x_2 ... is a sequence of independent observations from a Poisson distribution with mean λ. We want to test the null hypothesis $H_1: \lambda = \lambda_1$ against the alternative $H_2: \lambda = \lambda_2 (> \lambda_1)$ with strength (α_1, α_2).

For any positive integer m, the joint density function of X_m under H_i is

$$(1.2.1) \qquad f_{im} = \lambda_i^{\sum_{i=1}^{m} x_i} e^{-m\lambda_i} \Big/ \prod_{i=1}^{m} x_i \,!$$

Hence,

$$(1.2.2) \qquad f_{2m}/f_{1m} = (\lambda_2/\lambda_1)^{\Sigma x_i} e^{-m(\lambda_2-\lambda_1)}$$

and

$$(1.2.3) \qquad z_i = \log \; f(x_i:\lambda_2)/f(x_i:\lambda_1) \; = x_i \; \log(\lambda_2/\lambda_1) - (\lambda_2-\lambda_1)$$

Using Wald's values (1.1.4) for A and B and taking logarithms of (1.1.2) and (1.1.3), the acceptance rules are given as follows:

At the m-th stage of the experiment,

(i) stop sampling and accept H_1 if

$$\sum_{i=1}^{m} x_i \log(\lambda_2/\lambda_1) - m(\lambda_2-\lambda_1) \leq \log B' \tag{1.2.4}$$

ii) stop sampling and accept H_2 if

$$\sum_{i=1}^{m} x_i \log(\lambda_2/\lambda_1) - m(\lambda_2 - \lambda_1) \geq \log A' \tag{1.2.5}$$

iii) otherwise, continue experimentation.

(1.2.4) and (1.2.5) are conveniently written as

$$\sum_{i=1}^{m} x \leq \frac{\log\left\{\alpha_2/(1-\alpha_1)\right\}}{\log \lambda_2 - \log \lambda_1} + \frac{(\lambda_2-\lambda_1)m}{\log \lambda_2 - \log \lambda_1} \tag{1.2.6}$$

$$\equiv c_1 + s\,m \equiv a_m \quad \text{(say)}$$

$$\sum_{i=1}^{m} x \geq \frac{\log\left\{(1-\alpha_2)/\alpha_1\right\}}{\log \lambda_2 - \log \lambda_1} + \frac{(\lambda_2 - \lambda_1)\,m}{\log \lambda_2 - \log \lambda_1} \tag{1.2.7}$$

$$\equiv c_2 + s\,m \equiv b_m \text{ (say)} .$$

Thus, it is seen that the acceptance lines are parallel straight lines with the same slope $s = (\lambda_2 - \lambda_1)/(\log \lambda_2 - \log \lambda_1)$ which is independent of the desired error probabilities. However, c_1 and c_2 are determined by the hypotheses values and the error probabilities. If $\alpha_1 = \alpha_2 = \alpha$, from (1.2.6) and (1.2.7) it follows immediately that $c_2 = -c_1 = c$ (say).

1.3 OC and ASN Functions for the Poisson SPRT.

i) To obtain the OC function of the Poisson SPRT we have to find that value of $h(\lambda)$ (see (1.1.7) for which

$$(1.3.1) \qquad \sum_{x=0}^{\infty} \left[(\lambda_2/\lambda_1)^x \; e^{-(\lambda_2-\lambda_1)} \right]^{h(\lambda)} \lambda^x \; e^{-\lambda}/ x! = 1.$$

Summing the series and taking logarithms, (1.3.1) leads to

$$(1.3.2) \qquad \lambda(\lambda_2/\lambda_1)^{h(\lambda)} - (\lambda_2-\lambda_1)\, h(\lambda) - \lambda = 0.$$

Then, from (1.1.6), the OC function $L(\lambda)$ is approximately given by

$$(1.3.3) \qquad L(\lambda) \sim \left\{ \left(\frac{1-\alpha_2}{\alpha_1}\right)^{h(\lambda)} - 1 \right\} \Big/ \left\{ \left(\frac{1-\alpha_2}{\alpha_1}\right)^{h(\lambda)} - \left(\frac{\alpha_2}{1-\alpha_1}\right)^{h(\lambda)} \right\}$$

However, it is impossible to explicitly solve (1.3.2) for h. The Statistical Research Group, Columbia University [10], gave an indirect method of obtaining a solution, but even this method does not yield the OC function directly for a given value of λ.

For practical purposes, we can find the approximate values of the OC function for the following particular values of λ to give a rough picture of the OC function of the test.

	λ	$L(\lambda)$
(1.3.4)	0	1
(1.3.5)	λ_1	$1 - \alpha_1$
(1.3.6)	s	$c_2/(c_2 - c_1) = 1/2$ if $\alpha_1 = \alpha_2 = \alpha$
(1.3.7)	λ_2	α_2
(1.3.8)	∞	0

Since for $\lambda = s$ we have $h(\lambda) = 0$ as the only solution of (1.3.2), (by L'Hospital's rule) (1.3.6) is obtained by taking the limit value of (1.1.6).

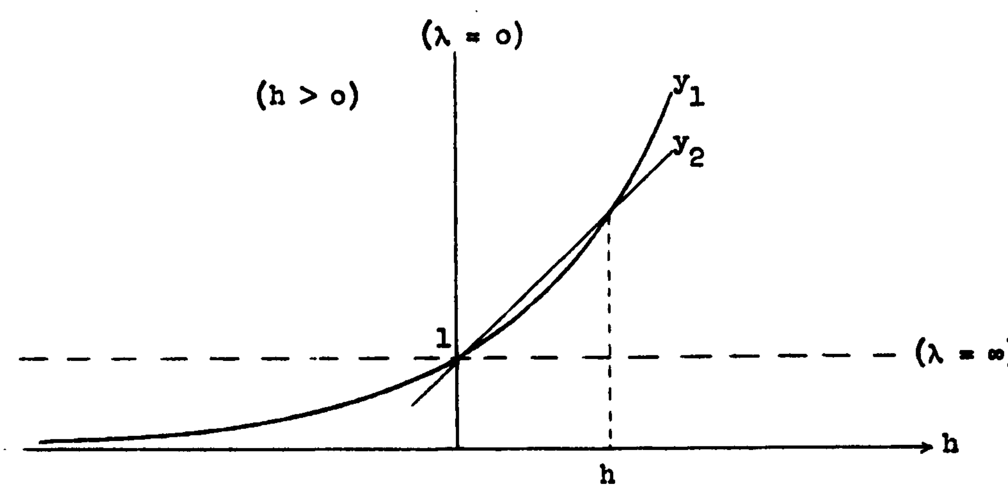
We shall introduce a graphical solution for (1.3.2). By (1.3.2)

$$(1.3.9) \qquad e^{h \log(\lambda_2/\lambda_1)} = 1 + (\lambda_2 - \lambda_1)\, h/\lambda \ .$$

To solve (1.3.9) for h, make the following transformations:

$$(1.3.10) \qquad y_1 = e^{h \log(\lambda_2/\lambda_1)} \ , \quad y_2 = 1 + (\lambda_2 - \lambda_1)\, h/\lambda \ .$$

Since $\log(\lambda_2/\lambda_1)$ and $(\lambda_2 - \lambda_1)/\lambda$ are fixed numbers for any given λ, y_1 is an exponential curve, and y_2 is a straight line. Then the solution of (1.3.2) can be found as the intercept of the straight line y_1 and the exponential curve y_2 as shown below .



The accuracy of the graphical solution can be improved, if desired, by a Newton iterative procedure.

From the h found by this method we can obtain the approximation of $L(\lambda)$ by substituting h into (1.1.6).

ii) In order to obtain the ASN function from (1.1.8) or (1.1.9) we have to know the OC function, $L(\lambda)$. However, as we have seen in the preceding parts, the OC function can only be found approximately. For the following particular values, the following approximations to the ASN function may be found from (1.1.8), (1.1.9) and (1.3.4) to (1.3.8):

	λ	$E_\lambda(n)$
(1.3.11)	0	$-c_1/s$
(1.3.12)	λ_1	$\left\{(1-\alpha_1)\,c_1+\alpha_1 c_2\right\}/(\lambda_1 - s)$
		$=(1-2\alpha)c/(s-\lambda_1)$ if $\alpha_1=\alpha_2=\alpha$
(1.3.13)	s	$-c_1c_2/s \;=\; c^2/s$ if $\alpha_1=\alpha_2=\alpha$
(1.3.14)	λ_2	$\left\{(1-\alpha_2)\,c_2+\alpha_2 c_1\right\}/(\lambda_2 - s)$
		$=(1-2\alpha)c/(\lambda_2-s)$ if $\alpha_1=\alpha_2=\alpha$.
(1.3.15)	∞	θ

(1.3.13) is obtained from (1.1.9) since $E_s(z)=0$. These particular five values of λ will serve sufficiently well for most practical purposes.

Although it is known that $E_\lambda(n^2)$ exists, neither exact nor simple approximate formulae to obtain the variance of N have yet been found, and very few empirical studies have been undertaken. However, from Tables VII and VIII it will be seen that the standard deviation of N for a truncated SPRT tends to be very large when λ is between λ_1 and λ_2 and α is very small.

1.4. The Truncated SPRT

Since the SPRT has no upper bound for the sample size, for practical purposes we may have to force the test to terminate at some definite stage, say n_o, of the experiment. Wald [11] gave these rules for truncation.

If the test does not terminate by the n_o-th stage, at the n_o-th stage,

i) stop sampling and accept H_1 if

$$\text{(1.4.1)} \qquad \log B < \sum_{i=1}^{n_o-1} z_i < \log A \quad \text{and} \quad \log B < \sum_{i=1}^{n_o} z_i < 0$$

ii) stop sampling and accept H_2 if

$$\text{(1.4.2)} \qquad \log B < \sum_{i=1}^{n_o-1} z_i < \log A \text{ and } 0 < \sum_{i=1}^{n_o} z_i < \log A$$

In this thesis we use the following rules for truncating the SPRT. Let n_o be the maximum sample size of the MPRT (see Chapter II) under the same hypothesis and strength, then the terminal decision at the n_o-th stage will be the following:

i) Stop sampling and accept H_1 if

$$\sum_{i=1}^{n_o} x_i < \frac{a_{n_o} + b_{n_o}}{2}$$

ii) Stop sampling and accept H_2 if

$$\sum_{i=1}^{n_o} x_i \geq \frac{a_{n_o} + b_{n_o}}{2}$$

If the SPRT is truncated at a sufficiently large n_o, it is to be expected that the error probabilities will not be greatly affected. However, truncation will reduce the average sample size somewhat, especially for intermediate λ values and it will have a similar effect on the SDN function.

CHAPTER II

MINIMUM PROBABILITY RATIO TEST AND ITS APPLICATION TO THE POISSON DISTRIBUTION

The expected value of the sample size (ASN) of a sequential test depends on both the parameter point θ and the particular sequential test. Ideally, we would like to find a sequential test which minimizes the ASN function for all values of θ, but no such "uniformly most economical" test exists. Hoeffding [5] has given a lower bound on the ASN at intermediate λ-values (Chapter III) for any sequential test meeting specified bounds on the error probabilities. In order to come close to achieving this lower bound under certain conditions, Hall [4] introduced the sequential minimum probability ratio test (MPRT). In this chapter we give a general discussion of this test and then consider the special case of the Poisson distribution.

2.1. The Minimum Probability Ratio Test (MPRT) for Testing a Simple Hypothesis against a Simple Alternative.

Suppose $x_1, x_2 \ldots$ is a sequence of independent observations with a common density function $f(x:\theta)$ and consider testing the null hypothesis H_1: $\theta = \theta_1$ against the alternative H_2: $\theta = \theta_2$. Let θ_0 be a parameter point between θ_1 and θ_2 and $f_i = f(x:\theta_i)(i=0.1.2)$. Denote by $X_m = (x_1, x_2 \ldots x_m)$ a sample of size m and f_{im} the joint density function of X_m.

We introduce the weight functions k_1 and k_2 where $k_1 + k_2 = 1$, for f_1 and f_2, respectively, to obtain a specified weighted average, α, of the error probabilities (see section 2.2).

The MPRT procedure is described as follows: at the m-th stage of the experiment,

i) stop sampling and accept H_1 if

(2.1.1) $k_2 f_{2m}/f_{om} \leq \alpha$ and $k_2 f_{2m}/k_1 f_{1m} \leq 1$

ii) stop sampling and accept H_2 if

(2.1.2) $k_1 f_{1m}/f_{om} \leq \alpha$ and $k_1 f_{1m}/k_2 f_{2m} \leq 1$

iii) otherwise, continue sampling.

Since the acceptance lines generally converge at some stage of the experiment, say n_o, the test has the upper bound n_o to the maximum number of trials, and therefore, either category i) or ii) will occur for some $m \leq n_o$.

2.2. Major Property of the MPRT

Let $P_i(d_j) = \alpha_i'$ be the probability of making the decision $d_j (j = 1, 2)$ when $H_i (i = 1,2$ and $i \neq j)$ is true. Then,

$$(2.2.1) \qquad k_1 \alpha_1' = k_1 P_1 (d_2) = k_1 \sum_{m=1}^{\infty} \int_{S_m^2} f_{1m}$$

$$(2.2.2) \qquad k_2 \alpha_2' = k_2 P_2 (d_1) = k_2 \sum_{m=1}^{\infty} \int_{S_m^1} f_{2m}$$

where S_m^i ($i = 1.2$) is the subsets of the sample space of X_m for making decision d_i, by (2.1.1) and (2.1.2), respectively.

We show one of the important properties of the MPRT by the following lemma:

Lemma 1 [4]:

For the MPRT with specified θ_o, the specified weighted average of the true error probabilities is not greater than the preassigned value α.

Proof: In S_m^2 it is obvious that

$$\min_{i=1.2} k_i f_{im} = k_1 f_{1m} \leq \alpha f_{om} \quad ;$$

that is,

$$(2.2.3) \qquad k_1 \alpha_1' \leq \alpha \sum_{i=1}^{\infty} \int_{S_m^2} f_{om} = \alpha P_o (d_2)$$

where $P_o(d_i)$ is the probability of d_i when θ_o is the true parameter value. Similarly, in S_m^1

$$(2.2.4) \qquad k_2' \alpha_2' \leq \alpha_o P_o(d_1)$$

Hence, it follows immediately that

$$(2.2.5) \qquad k_1 \alpha_1' + k_2 \alpha_2' \leq \alpha [P_o(d_1) + P_o(d_2)] = \alpha$$

assuming the acceptance lines converge.

For the case $k_1 = k_2 = 1/2$, from (2.2.3) to (2.2.5) we have $\alpha_1' \leq 2 \alpha P_o(d_1)$, $\alpha_2' \leq 2 \alpha P_o(d_2)$ and $\alpha_1' + \alpha_2' \leq 2 \alpha$. If θ_o can be so chosen that $P_o(d_1) = P_o(d_2) = 1/2$, then $\alpha_1' \leq \alpha$ and $\alpha_2' \leq \alpha$. In any case, at least one of the inequalities must hold since $\alpha_1' + \alpha_2' \leq 2\alpha$.

No method of choosing θ_o to achieve specified individual error probabilities is known, except for a limited result in the case of symmetry [4]. Such symmetry does not obtain in the Poisson case considered here. We shall attempt to achieve equal error probabilities by a method described in section 2.4.

2.3. The MPRT for the Poisson Distribution

Let $x_1, x_2 \ldots$ be a sequence of independent observations from a Poisson distribution with mean λ. We wish to test the null hypothesis H_1: $\lambda = \lambda_1$ against the alternative H_2: $\lambda = \lambda_2$ ($> \lambda_1$), and assume weight functions $k_1 = k_2 = 1/2$ and $\alpha < 1/2$.

The MPRT is carried out as follows:

at the m-th stage of the experiment,

i) stop sampling and accept H_1 if

$$f_{2m}/f_{1m} = \frac{\prod_{i=1}^{m} \lambda_2^{x_i} e^{-\lambda_2}/x_i!}{\prod_{i=1}^{m} \lambda_o^{x_i} e^{-\lambda_1}/x_i!} \leq 2\alpha$$

i.e.

$$\sum_{i=1}^{m} x_i \leq \frac{\log 2\alpha}{\log \lambda_2 - \log \lambda_o} + \frac{m(\lambda_2 - \lambda_o)}{\log \lambda_2 - \log \lambda_o} \equiv c_1 + r_1 m \text{ (say)} \tag{2.3.1}$$

which implies $f_{2m} \leq f_{1m}$

ii) stop sampling and accept H_2 if

$$(2.3.2)\quad \sum_{i=1}^{m} x_i \geq \frac{-\log 2\alpha}{\log \lambda_o - \log \lambda_1} + \frac{m(\lambda_o - \lambda_1)}{\log \lambda_o - \log \lambda_1} \equiv c_2 + r_2 m \text{ (say)}$$

which implies $f_{2m} \geq f_{1m}$

iii) otherwise, continue sampling.

(2.3.1) and (2.3.2) will define the acceptance lines of the test. These two lines meet at $m = n_o'$ where

$$n_o' = \frac{c_1 - c_2}{r_2 - r_1}$$

and hence, this is the upper bound for the sample size. The actual maximum sample number n_o will be an integer slightly smaller than n_o'.

An illustrative diagram for carrying out this test with a comparison of the truncated SPRT and the FSST appears in Figure 1 in the Appendix.

2.4. A Choice for θ_o

In the MPRT it is possible to choose θ_o at any parameter point other than θ_1 and θ_2, depending on our emphasis on the error probabilities α_1' and α_2' and the weight functions k_1 and k_2. In this thesis, we will choose θ_o in an attempt to obtain approximately equal error probabilities at θ_o, i.e. $P_o(d_1) = P_o(d_2)$. From this point of view, we take θ_o as the value which yields equal "divergence" [7] between θ_1 and θ_o, and between θ_o and θ_2.

Suppose the null hypothesis is H_1: $f(x) = f_1(x)$ and the alternative H_2: $f(x) = f_2(x)$; then for a given x, the mean information for discrimination per observation in favor of H_1 against H_2 is defined [7] as

$$I\,(1:2) = \int_{\mathcal{X}} f_1(x) \log \frac{f_1(x)}{f_2(x)}\, d\,x$$

where $\mathcal{X}$ is the entire sample space. $I\,(2:1)$ is similarly defined. Under the assumption that the probability measures for $f_1(x)$ and $f_2(x)$ are "absolutely continuous", the integrals always exist.

The divergence $D(1,2)$ between H_1 and H_2 is defined [7] as

$$(2.4.1) \quad D(1,2) = I(1:2) + I(2:1) = \int_{\mathcal{X}} [f_1(x) - f_2(x)] \log \frac{f_1(x)}{f_2(x)}\, d\,x.$$

$D(1,2)$ is a measure of the difficulty of discriminating between hypotheses H_1 and H_2, and thus will be a reasonable choice to determine θ_o for the MPRT. Further detailed discussion concerning information and divergence will be found in Kullback [7].

Example: Poisson case

Suppose H_1: $\lambda = \lambda_1$ i.e. $f_1(x) = \lambda_1^x \; e^{-\lambda_1}/x!$

and

$$H_2: \; \lambda = \lambda_2 \;\; \text{i.e.} \;\; f_2(x) = \lambda_1^x \; e^{-\lambda_1}/x!$$

Then, by definition

$$I(1:2) = \sum_{x=o}^{\infty} \frac{\lambda_1^x}{x!} \; e^{-\lambda_1} \; \log \frac{\lambda_1^x \; e^{-\lambda_1}/x!}{\lambda_2^x \; e^{-\lambda_2}/x!}$$

$$= \sum_{x=o}^{\infty} \frac{\lambda_1^x}{x!} \; e^{-\lambda_1} \;\; x(\log \lambda_1 - \log \lambda_2) - (\lambda_1 - \lambda_2)$$

$$(2.4.2) \qquad = (\log \lambda_1 - \log \lambda_2)\,\lambda_1 - (\lambda_1 - \lambda_2)$$

Similarly,

$$(2.4.3) \quad I(2:1) = (\log \lambda_2 - \log \lambda_1)\,\lambda_2 - (\lambda_2 - \lambda_1)$$

Hence, from (2.4.1) to (2.4.3),

$$(2.4.4) \quad D(1,2) = (\lambda_2 - \lambda_1)(\log \lambda_2 - \log \lambda_1)$$

For some λ_o, where $\lambda_1 < \lambda_o < \lambda_2$, the difficulty of discriminating between hypotheses λ_1 and λ_o and between λ_o and λ_2 is equal. That is, λ_o is obtained from

$D(1,0) = D(0,2)$; that is

$$(\lambda_o - \lambda_1)(\log \lambda_o - \log \lambda_1) = (\lambda_2 - \lambda_o)(\log \lambda_2 - \log \lambda_o)$$

$$(2.4.5) \quad g(\lambda_o) \equiv (\log \lambda_2 - \log \lambda_1)\lambda_o + (\lambda_2 - \lambda_1)\log \lambda_o + (\lambda_1 \log \lambda_1 - \lambda_2 \log \lambda_2) = 0 .$$

Newton's approximation method was used to solve this equation numerically, starting with the initial value $\lambda_{o.o} = (\lambda_1 + \lambda_2)/2$. That is,

$$\lambda_{o.i+1} = \lambda_{o.i} + g(\lambda_o) \Big/ \frac{d}{d\lambda_o} g(\lambda_o)$$

$$(2.4.6) \qquad = \lambda_o + \frac{\lambda_o(\log \lambda_2 - \log \lambda_1) + (\lambda_2 - \lambda_1)\log \lambda_o + \lambda_1 \log \lambda_1 - \lambda_2 \log \lambda_2}{\log \lambda_2 - \log \lambda_1 + (\lambda_2 - \lambda_1)/\lambda_o}$$

$$(i = 0.1. \ \ldots)$$

with the criterion for stopping the iteration

$$\left| \lambda_{o.i+1} - \lambda_{o.i} \right| \leq 10^{-5}$$

and $i \leq 200$.

By this method, divergence values in the Appendix were computer calculated. This λ_o value is there denoted by D.

A few examples were considered using $\lambda_o = s$ instead of $\lambda_o = D$; a comparison of these choices is discussed in Chapter V. It was found that $D < s$.

CHAPTER III

FIXED SAMPLE SIZE TEST AND HOEFFDING'S LOWER BOUND FOR THE EXPECTED SAMPLE SIZE

3.1. Most Powerful Test for the Poisson Distribution.

To make a sample size comparison between the MPRT and a fixed sample size test (FSST), we need to find the minimum sample size for the most powerful FSST with specified bounds on the error probabilities. The most powerful FSST is obtained by the Neyman-Pearson lemma.

For the Poisson case with sample size m, the test is carried out as follows:

i) accept H_2 if

$$(3.1.1) \qquad \frac{f_{2m}}{f_{1m}} = \frac{\prod_{i=1}^{m} \lambda_2^{x_i} e^{-\lambda_2}/x_i\,!}{\prod_{i=1}^{m} \lambda_1^{x_i} e^{-\lambda_1}/x_i\,!} \geq k\ (>0)$$

ii) otherwise, accept H_1 .

Taking logarithms on both sides of (3.1.1), it follows that

$$\sum_{i=1}^{m} x_i\,(\log \lambda_2 - \log \lambda_1) - n(\lambda_2 - \lambda_1) \geq k'$$

and

$$(3.1.2) \qquad \sum_{i=1}^{m} x_i \geq k''$$

We know that the random variable $y = \sum_{i=1}^{m} x_i$ has the Poisson distribution with mean $m\lambda$, i.e.

$$g(y) = (m\lambda)^y e^{-m\lambda}/y! \tag{3.1.3}$$

If the Type I error α_1 and the size of sample m are given, k'' is obtained as the smallest integer for which

$$1 - F_{m\lambda_1}(k'' - 1) \leq \alpha_1 \tag{3.1.4}$$

where $F_\lambda(x)$ is the Poisson distribution function at x. If the Type I error α_1 and the Type II error α_2 are preassigned, the sample size can be obtained by finding k'' and the minimal m which will satisfy,

$$1 - F_{m\lambda_1}(k'' - 1) \leq \alpha_1 \tag{3.1.5}$$

and

$$F_{m\lambda_2}(k'' - 1) \leq \alpha_2 \tag{3.1.6}$$

3.2. Normal Approximation for the Poisson FSST

If λ is large, the Poisson distribution can be approximated by the Normal distribution. That is,

$$F_\lambda(x) = \Phi\left(\frac{x - \lambda + 1/2}{\sqrt{\lambda}}\right) \tag{3.2.1}$$

where

$$\Phi(t) = \int_{-\infty}^{t} (1/\sqrt{2\pi})\, e^{-x^2/2}\, dx$$

Then, by (3.1.5) and (3.1.6)

$$\Phi\left(\frac{k'' - 1 - m\lambda_1 + 1/2}{\sqrt{m\lambda_1}}\right) \geq 1 - \alpha_1 \tag{3.2.2}$$

$$(3.2.3) \qquad \Phi\left(\frac{k'' - 1 - m\lambda_2 + 1/2}{\sqrt{m\lambda_2}}\right) \leq \alpha_2$$

are the required inequalities.

If the Poisson mean is not large, the normal approximation may not be adequate so that the following rule was used. If $(m\lambda_1)^{1/2} < 4.0$, several integer values were chosen in the neighborhood of the m which was obtained from the Normal approximation method. These were substituted into (3.1.5) and (3.1.6), with suitably chosen k'' s from the Poisson tables, until (3.1.5) and (3.1.6) were satisfied. The minimum such m is the sample size of the most powerful test. In the tables in Appendix those values obtained by the Normal approximation appear with *, such as n_F^* and $FSST^*$.

3.3. Hoeffding's Lower Bound (HLB) [5]

Let $x_1, x_2 \ldots$ be a sequence of independent observations with a common density function $f(x:\theta)$, and suppose we want to test the null hypothesis $H_1: \theta = \theta_1$ against the alternative $H_2: \theta = \theta_2$. Also, let $f(x:\theta) = f_i(x:\theta)$ under $H_i (i = 1.2)$, and the decision d_i is to accept H_i.

As one of the optimum properties of the SPRT, it has been proved [12] that the SPRT minimizes the expected sample size at the points θ_1 and θ_2 subject to specified bounds on the error probabilities. In general, its expected sample size is largest when θ is between θ_1 and θ_2 but not always (see Figure 2). However, there exist tests which have a smaller expected sample size at intermediate θ values than the SPRT. Kiefer and Weiss [6] have discussed important qualitative properties of such tests. These tests may be judged by comparing, at any parameter point θ, the expected sample size of the test with the smallest expected sample size attainable by any test having the same error probabilities at θ_1 and θ_2.

The Hoeffding Lower Bound on the expected sample size at any θ_o is given by

$$(3.3.1) \quad E_{\theta_o}(N) \geq \left\{ [(\tau/4)^2 - \zeta \log (\alpha_1 + \alpha_2)]^{1/2} - \tau/4 \right\}^2 / \zeta^2$$

assuming that the following integrals exist

$$(3.3.2) \qquad \zeta = \max_{i=1,2} (\zeta_1, \zeta_2) \; , \quad \zeta_i = \int f_o \log (f_o/f_i) \; d\mu$$

and

$$(3.3.3) \qquad \tau^2 = \int \left\{ \log (f_2/f_1) - \zeta_1 + \zeta_2 \right\}^2 f_o \; d\mu \; .$$

Also assume that

$$f_o(x) = 0 \text{ implies } \min_{i=1,2} f_i = 0$$

and that

$$E_o \left(\sum_{j=1}^{N} Y_j\right)^2 = \tau^2 \; E_o(N)$$

is satisfied where $Y_j = \log \left\{ f_2(x)/f_1(x) \right\} - \zeta_1 + \zeta_2$.

Hoeffding proved [5] that for all θ_o, (3.3.1) gives a lower bound among all strength $(\alpha_1 . \alpha_2)$ tests.

For the Normal density function with variance one and mean θ_i where $\theta_o = 0$, $\theta_1 = -\delta$ and $\theta_2 = \delta > 0$, he compared the numerical values obtained by his lower bound with those of the fixed sample size test, one of Anderson's tests [1] and the SPRT of the same strength (α_1, α_2). For $\alpha_1 = \alpha_2 = \alpha < 1/2$, the results indicate that equality in (3.3.1) is nearly obtainable with a FSST if α is very small and with the SPRT if α is sufficiently large. For other α values, those commonly used in practice, the MPRT (which coincides with one of Anderson's tests) nearly attains this bound.

3.4. Hoeffding's Lower Bound for the Poisson Distribution

Suppose $f(x:\lambda) = \lambda^x e^{-\lambda}/x\,!$ and we wish to test $H_1: \lambda = \lambda_1$ against $H_2: \lambda = \lambda_2\ (> \lambda_1)$ with strength of the test $\alpha_1 = \alpha_2 = \alpha < 1/2$. Then, by definition (3.3.2)

$$\zeta_1 = \sum_{x=0}^{\infty} \lambda_o^x \frac{e^{-\lambda_o}}{x!} \log \frac{\lambda_o^x e^{-\lambda_o}/x!}{\lambda_1^x e^{-\lambda_1}/x!}$$

$$= \left(\sum_{x=0}^{\infty} \lambda_o e^{-\lambda_o}/x! \right) x \log(\lambda_o/\lambda_1) - (\lambda_o - \lambda_1) \sum_{x=0}^{\infty} \lambda_o^x e^{-\lambda_o}/x!$$

$$= \lambda_o (\log \lambda_o - \log \lambda_1) - (\lambda_o - \lambda_1). \tag{3.4.1}$$

Therefore,

$$\zeta_1 = \lambda_o (\log \lambda_o - \log \lambda_1) - (\lambda_o - \lambda_1) \tag{3.4.2}$$

$$\zeta_2 = \lambda_o (\log \lambda_o - \log \lambda_2) - (\lambda_o - \lambda_2) \tag{3.4.3}$$

Also by definition (3.3.3)

$$\tau^2 = \sum_{x=0}^{\infty} \frac{\lambda_o e^{-\lambda_o}}{x!} \left\{ \log \frac{\lambda_2^x e^{-\lambda_o}}{\lambda_1^x e^{-\lambda_1}} - (\log \lambda_o - \log \lambda_1) + (\lambda_o - \lambda_1) \right.$$

$$\left. + (\log \lambda_o - \log \lambda_2) + (\lambda_2 - \lambda_o) \right\}^2$$

$$= \sum_{x=0}^{\infty} \frac{\lambda_0 e^{-\lambda_0}}{x\,!} \left\{ x(\log \lambda_2 - \log \lambda_1) + \lambda_0(\log \lambda_1 - \log \lambda_2) \right\}^2$$

$$= \sum_{x=0}^{\infty} (\lambda_0^x \; e^{-\lambda_0}/x\,!) \left\{(\log \lambda_2 - \log \lambda_1)^2 \, (x - \lambda_0)^2 \right\}$$

$$(3.4.4) \quad = \lambda_0 \, (\log \lambda_2 - \log \lambda_1)^2$$

For any λ_0 we can obtain Hoeffding's lower bound by substituting (3.4.2) or (3.4.3) and (3.4.4) into (3.3.1).

From (3.4.2) and (3.4.3) it is easily seen that the relation $\zeta_1 = \zeta_2 = \zeta$ is obtained at

$$\lambda_0 = (\lambda_2 - \lambda_1)/(\log \lambda_2 - \log \lambda_1)$$

and this value of λ_0 is equal to s, the common slope of the acceptance lines of the SPRT for the Poisson distribution under the same hypothesis. For $\lambda_0 = s$, Hoeffding's lower bound is given by

$$(3.4.5) \quad E_s(N) \geq \frac{\left[\left\{\frac{s}{16}\left(\log \frac{\lambda_2}{\lambda_1}\right)^2 - \left(s \log \frac{s}{\lambda_1} - s + \lambda_1\right)\log 2\alpha\right\}^{1/2} - \frac{s}{4}^{1/2} \log \frac{\lambda_2}{\lambda_1}\right]^2}{s \log \frac{s}{\lambda_1} - s - \lambda_1}$$

The $E_s(N)$ of the SPRT is given approximately by (1.3.13), therefore, we can compare Hoeffding's lower bound with the approximation of the SPRT ASN at $\lambda_0 = s$.

The HLB's in the tables in the Appendix were computer calculated.

CHAPTER IV

THE PROGRAMS FOR THE MPRT AND THE $SPRT_o$ FOR THE POISSON DISTRIBUTION

In this Chapter we discuss the programs by which the exact OC, ASN and SDN functions of the MPRT and $SPRT_o$ were obtained for the Poisson distribution. The programs (for the MPRT and the $SPRT_o$) were written in the "IT" language [9] and stored as "K. Fukushima, MPRT - A" and "K. Kukushima, SPRT - A", respectively, in the library of the Research Computation Center, the consolidated University of North Carolina, Chapel Hill, North Carolina, for future use. The detailed compiler program for the MPRT is shown in A.7.

4.1. Brief Explanation of the Program for the MPRT

Let

$\underline{m}_i$ (integer): the acceptance boundary for the i-th trial.

$\bar{m}_i$ (integer): the rejection boundary for the i-th trial.

$p(\underline{m}_i)$: the probability of acceptance at the i-th trial.

$p(\bar{m}_i)$: the probability of rejection at the i-th trial.

$p(m_{i.j})$: the probability of j defects ($j = \Sigma x$) at the i-th trial.

n_o': the point at which the acceptance and rejection lines intersect.

n_o: the maximum possible number of trials; the least i such that

$$\bar{m}_i - \underline{m}_i \leq 1$$

n_1: the number of Poisson probabilities to be calculated.

$$n_1 = c_2 - c_1 + 5$$

a_1, a_2: the ordinates (Σ x) of the acceptance and rejection lines at n'_o .

λ 1, λ 2, ... λ 11 (input): The parameter points for which the OC, ASN and SDN functions are to be calculated.

By (1.2.6) and (1.2.7), after we determine the boundaries for acceptance and rejection and n_o, we perform the following calculations for $(i \leq n_o)$

$$P(m_{i.j}) = \sum_{k=\underline{m}_{i-1}+1}^{\min.(j,\overline{m}_{i-1}-1)} P(m_{i-1.k}) \cdot P_\lambda(j - k)$$

where $P_\lambda(x)$ is the Poisson probability of x with mean λ.

$$P(\underline{m}_i) = \sum_{k=\underline{m}_{i-1}+1}^{\underline{m}_i} \left\{ P(m_{i-1.k}) \cdot \sum_{x=o}^{\underline{m}_i-k} P_\lambda(x) \right\} \qquad \text{if } \underline{m}_{i-1} \neq \underline{m}_i$$

$$P(\underline{m}_i) = 0 \qquad \text{otherwise}$$

and

$$P(\overline{m}_i) = \sum_{k=\underline{m}_{i-1}+1}^{\overline{m}_{i-1}-1} \left\{ P(m_{i-1.k}) \sum_{x=\overline{m}_i-k}^{\infty} P_\lambda(x) \right\}$$

The OC, ASN and SDN functions are given by

$$L(\lambda) = \sum_{i=1}^{n_o} P(\underline{m}_i)$$

$$E_\lambda(N) = \sum_{i=1}^{n_o} i \left\{ P(\underline{m}_i) + P(\overline{m}_i) \right\}$$

ERRATA PAGE

for

A COMPARISON OF SEQUENTIAL TESTS FOR THE POISSON PARAMETER

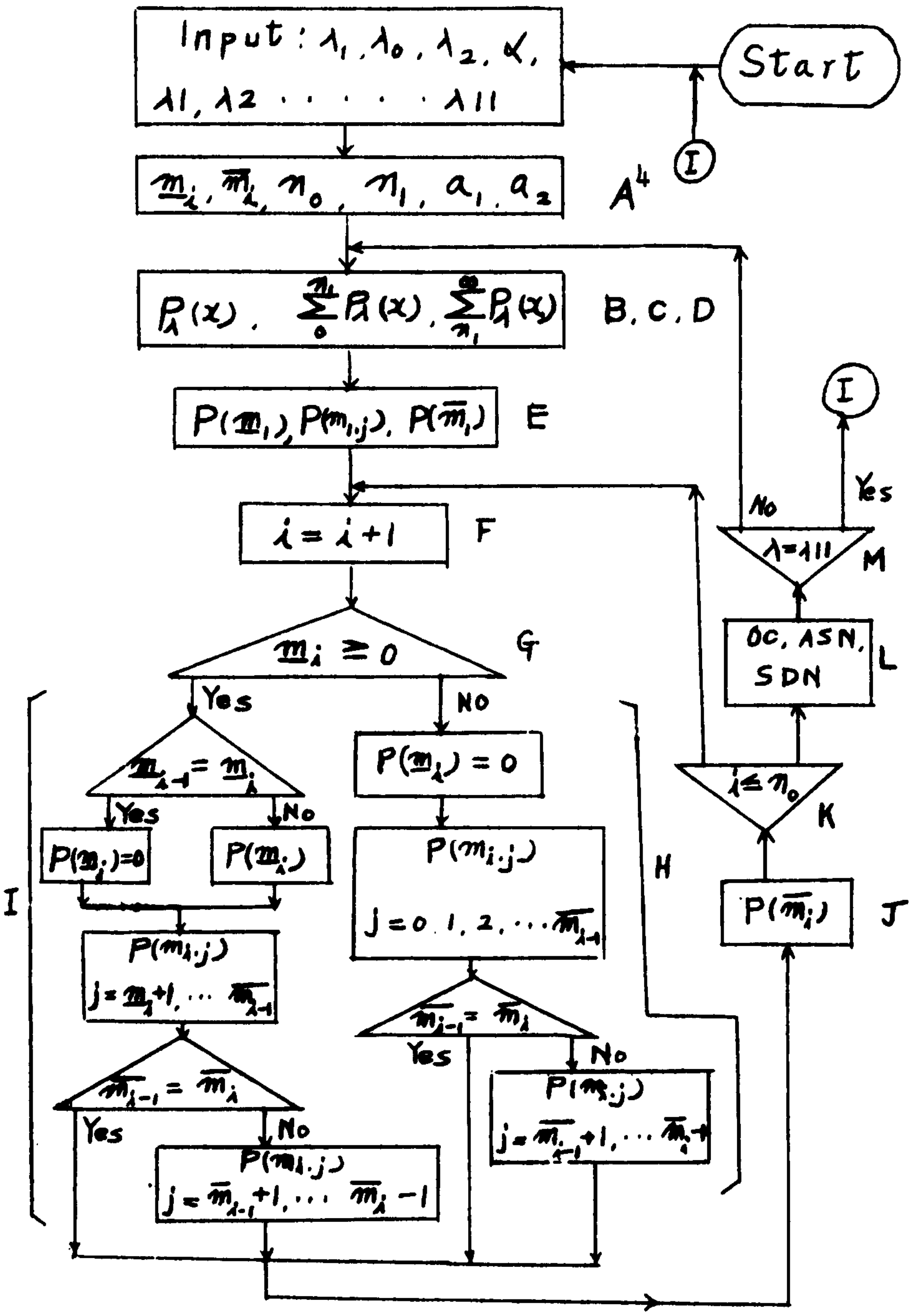
by

Kozo Fukushima

Page	Line number from the top	Misprint	Correction
7	7	$/E_\theta(z)$ if $E_\theta(z) = 0$	$/E_\theta(z^2)$ if $E(z) = 0$
15	7	$k_2 f_{2m}/k_1 f_{1m} \leq 1$	$k_2 f_{2m}/k_1 f_{1m} < 1$
17	17	$f_{2m}/f_{1m} = \dfrac{\prod_{i=1}^{m} \lambda_2^{x_i} e^{-\lambda_2}/x_i!}{\prod_{i=1}^{m} \lambda_0^{x_i} e^{-\lambda_2}/x_i!}$	$f_{2m}/f_{om} = \dfrac{\prod_{i=1}^{m} \lambda_2^{x_i} e^{-\lambda_2}/x_i!}{\prod_{i=1}^{m} \lambda_0^{x_i} e^{-\lambda_0}/x_i!}$
17	20	which implies $f_{2m} \leq f_{1m}$	which implies $f_{2m} < f_{1m}$
19	16	i.e. $f_2(x) = \lambda_1^x e^{-\lambda_1}/x!$	i.e. $f_2(x) = \lambda_2^x e^{-\lambda_2}/x!$
20	16	$\lambda_{o.i+1} = \lambda_{o.i} + g(\lambda_o)/ \frac{d}{d\lambda_o} g(\lambda_o)$	$\lambda_{o.i+1} = \lambda_{o.i} - g(\lambda_o)/ \frac{d}{d\lambda_o} g(\lambda_o)$
20	17	(2.4.6) $= \lambda_o +$	(2.4.6) $= \lambda_o -$
33	20	The time for computation	The time for compilation
36	16	the MPRT is uniformly	the SUN of the MPRT is uniformly
53	5	$(n_{os} = 11)$	$(n_{os} = 17)$
56	2	$(n_{os} = 571)$	$(n_{os} = 71)$
67	17	.00071 .00034 .00023	.0000071 .000034 .000023

$$SDN = \left[\sum_{i=1}^{n_o} i^2 \left\{ P(\underline{m}_i) + P(\overline{m}_i) \right\} - \left\{ E_\lambda(N) \right\}^2 \right]^{1/2}$$

The overall flow diagram is shown on the following page.



4. The steps A, B ... M appear in the compiler program in A.7 correspondingly.

4.2. Variable Assignments for "MPRT - A"

$Y\ 1$ (input) $= \lambda_1$, $Y\ 2$ (input) $= \lambda_o$ $Y\ 3$ (input) $= \lambda_2$

$Y\ 4$ (input) $= \alpha$, $Y\ 28$, $Y\ 29 \ldots Y\ 38$ (input) $= \lambda\ 1, \lambda 2, \ldots \lambda\ 11$

$N\ 1 = i$, $N\ 2 = j$, $N\ 900 = n_o$, $N\ 902 = n_1$

$N\ 10$ — $N\ 500$: $\underline{m}_1$, $N\ 500$ — $N\ 904$: $\bar{m}_1$

$N\ 905$ — $N\ 1050$: Alphanumeric

$Y\ 5 = c_1$, $Y\ 6 = r_1$, $Y\ 7 = c_2$, $Y\ 8 = r_2$

$Y\ 100$ — $Y\ 499$: $P(\underline{m}_{1.j})$, $Y\ 500$ — $Y\ 999$: $P(\bar{m}_{1.j})$

$Y\ 1000$ — $Y\ 1999$: $P_\lambda(j)$, $Y\ 2000$ — $Y\ 2400$: $P(\underline{m}_1)$

$Z\ 0$ — $Z\ 999$: $\sum_{x=j}^{\infty} P_\lambda(x)$, $Z\ 1000$ — $Z\ 1999$: $\sum_{x=o}^{j} P_\lambda(x)$

$Z\ 2000$ — $Z\ 2400$: $P(\bar{m}_1)$ where $j = 0, 1, 2, \ldots n_1$

4.3. The Program for "SPRT - A"

The storage spaces for the variable assignments are very similar to the program for the MPRT, except for the following changes. For input,

i) $Y\ 1 = \lambda_1$, $Y\ 2 = \lambda_2$, $Y\ 3 = \alpha_1$, $Y\ 4 = \alpha_2$

ii) the truncation point ($N\ 900 = n_o$) must be given

iii) the number of parameter points for which the OC, ASN and SDN functions are to be calculated is 10(Y29, ... Y38) instead of 11 in "MPRT - A" .

4.5. Outputs and Capacities of the Programs, "MPRT - A" and "SPRT-A".

Unconditional outputs for these programs are as follows:

$\lambda_1, \lambda_o, \lambda_2, \alpha$ and $\lambda 1, \lambda 2 \ldots \lambda 11$

c_1, r_1, c_2, r_2, n_o' and n_o (for the MPRT - A)

c_1, s and c_2 (for the SPRT - A)

a_1 and a_2 (where $a_1 = a_2$)

For $\lambda = \lambda 1, \lambda 2, \ldots \lambda\ 11$

$$\sum_{i=1}^{n_o} P_\lambda(\underline{m}_i)\ , \qquad \sum_{i=1}^{n_o} P_\lambda(\overline{m}_i)$$

ASN, the variance of N, and SDN

For conditional outputs of the programs, the following outputs are added:

n_1 , $\underline{m}_i$ and $\overline{m}_i$ $(i = 0, 1, 2, \ldots n_o)$

$P_\lambda(j)$, $\sum_{x=0}^{j} P_\lambda(x)$ and $\sum_{x=j}^{\infty} P_\lambda(x)$ where $j = 0, 1, 2, \ldots\ n_1$

$P(m_{i.j})$ for the i-th trial $(i = 1.2 \ldots n_o)$

$P(\underline{m}_{n_o})$ and $P(\overline{m}_{n_o})$ (for SPRT - A)

By "MPRT-A" and "SPRT - A", we can take any hypothesis values λ_1 and λ_2 and the error probabilities $\alpha < \frac{1}{2}$ as long as n_o and $\Sigma\ x$ do not exceed 400 and 500, respectively. Also, the core storage used in the computer is 7695 for "MPRT - A" and 7752 for "SPRT - A". However, one can change the program according to the requirements of each particular problem and a computer capacity.

The time for computation was about three minutes, and the calculation of the OC, ASN and SDN functions at one parameter point for each of eight sets of hypotheses with four error levels; that is, one λ value for each of thirty-two test situations, took about fourteen minutes. Similar time is required for "SPRT-A".

CHAPTER V

SUMMARY

In this chapter we discuss the characteristics of the MPRT from the data obtained, with comparison of other tests, the $SPRT_o$, the SPRT and the FSST.

By the use of divergence to obtain λ_o, and $k_1 = k_2 = 1/2$, the test nearly achieves $P_o(d_1) = P_o(d_2)$ when α is not very small, and λ_1 and λ_2 are not very distinct. As already mentioned in section 3.2, the sum of the exact error probabilities, $\alpha_1' + \alpha_2'$, is smaller than the preassigned level 2α. Moreover, even though the test for the Poisson distribution is not symmetric, we observe that $\alpha_1' \leq \alpha_1$ and $\alpha_2' \leq \alpha_2$, and if α is not very small, $\alpha_1' \leq \alpha_2'$ in general. As mentioned in the introduction, the use of $\lambda_o = D$ gives slightly better results to achieve $P_o(d_1) = P_o(d_2)$ for large α values and $\lambda_o = S$ gives slightly better results for small α.

Comparing the OC functions of the MPRT with the $SPRT_o$ and FSST the following points may be found:

i) the $SPRT_o$ has generally higher discrimination than the MPRT between λ_1 and λ_2 for small α, the OC function of the MPRT tends to be close to the $SPRT_o$. However, the difference between the OC functions is not sufficiently large to be of particular importance; and

ii) the MPRT has generally higher discrimination for small α than the FSST except near λ_1 and λ_2. But if α is large, the FSST tends to have uniformly higher discrimination.

From the tables and graphs shown in the Appendix we see the following characteristics of the ASN function of the MPRT.

i) The smaller the α-value, the closer to s is the maximum value of the ASN of the MPRT.

ii) For small α, the ASN of the MPRT is smaller than both the $SPRT_o$ and SPRT for some values of λ between λ_1 and λ_2, but the maximum of the ASN for the MPRT and $SPRT_o$ are not very different.

iii) The ASN of the MPRT is uniformly smaller than for the FSST except for extremely small α values.

iv) The HLB at $\lambda = D$ (and $\lambda = s$) is nearly attained for large values of α by the MPRT. For smaller values of α, even though the ASN of the MPRT is not close to the HLB, it is closer than the ASN of the SPRT or the FSST.

We observe that for any α values the MPRT is uniformly smaller than for the $SPRT_o$ which is presumably smaller than for the SPRT.

Therefore, the MPRT under these conditions appears to be advantageous as compared with these other tests when the parameter point lies near the average of λ_1 and λ_2, and particularly when α is small. Even if the ASN of the MPRT is slightly larger than of the SPRT, the use of the MPRT may be recommended because of its smaller SDN. Further studies of the MPRT for various weight functions k_1 and k_2 should yield useful results.

APPENDIX I

A.1. Notation

The following notation is used in the tables and figures in the Appendix:

OC : the operating characteristic function

ASN : the average sample number function

SDN : the standard deviation of the sample size N

FSST : the most powerful fixed sample size test

HLB : Hoeffding's lower bound for the ASN

MPRT : the minimum probability ratio test ($\lambda_o = D$)
the numbers in parentheses in the tables V and VI were obtained by the MPRT ($\lambda_o = s$)

SPRT : the sequential probability ratio test

SPRT : the sequential probability ratio test
the numbers which have * in the $SPRT_o$ in the tables VII and VIII were obtained by Wald's approximation for the SPRT

$SPRT_o$: the sequential probability ratio test truncated at n_o

α : the specified bound on each error probability

D : λ_o value for which the divergence between λ_1 and λ_o equals the divergence between λ_o and λ_2

s : the slope of the SPRT acceptance lines

n_F : the sample size of the FSST obtained from the Poisson distribution (with linear interpolation in the tables)

n_F^* : the sample size of the FSST obtained from Normal approximation

n_o : the maximum sample size of the MPRT ($\lambda_o = D$)

n_{os} : the maximum sample size of the MPRT ($\lambda_o = s$)

n_D : the HLB at $\lambda = D$

n_s : the HLB at $\lambda = s$

APPENDIX II
SAMPLING PLAN
A.2. SAMPLING PLAN

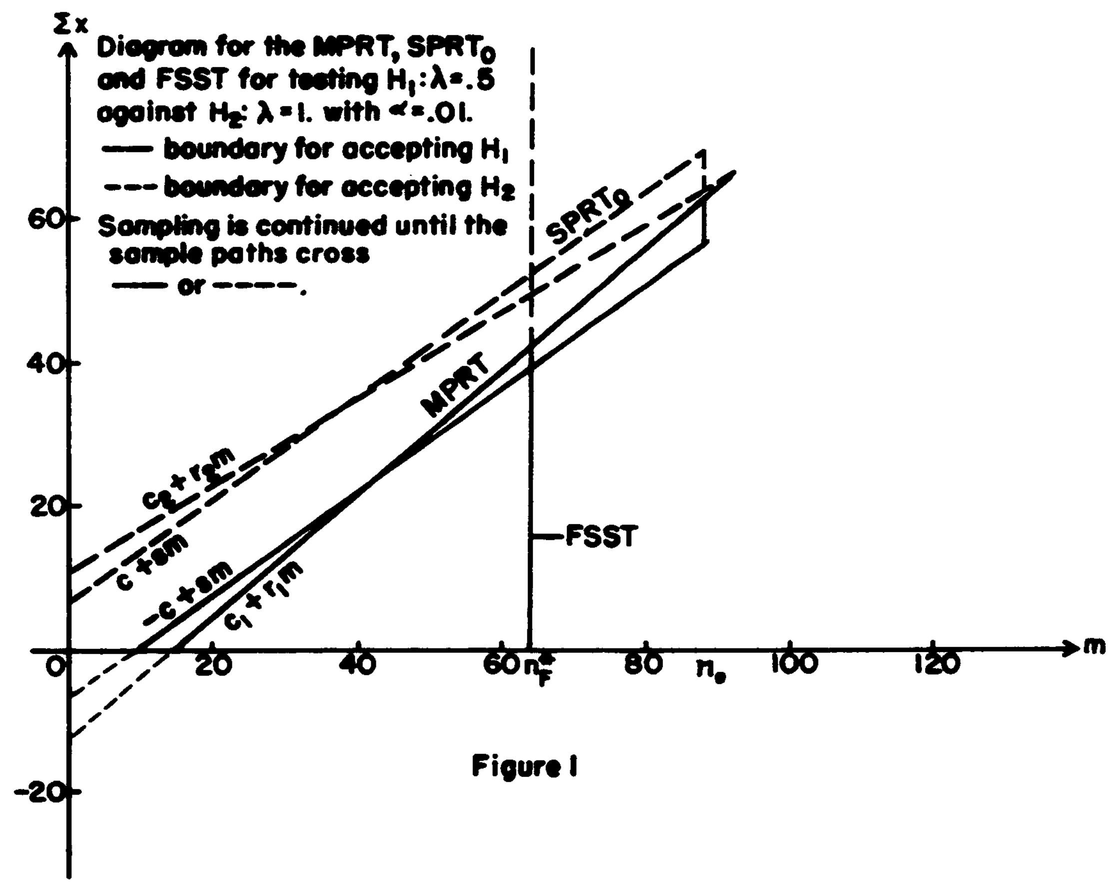


Figure I

APPENDIX III

A.3. TABLES I - IV: **Characteristics of the Poisson MPRT and SPRT Approximation.**

A.3.1. TABLES I - 1, 2, 3, 4.

H_1: λ = .1 against H_2: λ ; .3

s = .18205 D = .18652

- -

TABLE I - 1

α = .1, n_o = 52, n_F = 31

λ	OC		ASN		HLB	SDN of MPRT
	MPRT	SPRT*	MPRT	SPRT*		
0	1.	1.	15.	11.		0.
.1	.9203	.90	22.83	19.75		7.88
.17	.5854		24.63			10.37
s	.5175	.50	24.28	21.97	20.40	10.62
D	.4929		24.11		19.18	10.71
.25	.2112		20.31			10.97
.3	.0932	.10	16.90	13.73		10.13

TABLE I - 2

$\alpha = .05,\ n_o = 79,\ n_F = 52$

λ	OC		ASN		HLB	SDN of MPRT
	MPRT	SPRT*	MPRT	SPRT*		
0.	1.	1.	21.	15.		0.
.1	.9594	.95	34.15	29.48		11.07
.17	.5971		39.81			15.09
s	.5116	.50	39.28	39.46	34.37	15.55
D	.4805		38.98		32.17	15.71
.25	.1497		31.32			15.98
.3	.0455	.05	24.79	20.51		13.99

TABLE I - 3

$\alpha = .01,\ n_o = 140,\ n_F = 102$

λ	OC		ASN		HLB	SDN of MPRT
	MPRT	SPRT*	MPRT	SPRT*		
0	1.	1.	35.	23.		0.
.1	.9916	.99	59.68	49.96		16.65
.17	.6237		79.49			25.35
s	.5054	.50	78.59	96.10	71.46	26.47
D	.4621		77.88		66.48	26.87
.25	.0706		56.85			26.75
.3	.0087	.01	41.79	34.76		20.79

TABLE I - 4

$\alpha = .001,\ n_o = 223,\ n^*_F = 180$

λ	OC		ASN		HLB	SDN of MPRT
	MPRT	SPRT*	MPRT	SPRT*		
0	1.	1.	55.	35.		0.
.1	.9991	.999	95.04	76.47		22.04
.17	.6570		142.18			38.03
s	.5028	.50	141.1	217.1	130.8	40.22
D	.4459		139.7		121.1	41.13
.25	.0256		92.25			39.04
.3	.00088	.001	65.01	53.19		27.11

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A.3.2. TABLES II - 1. 2. 3. 4.

$H_1: \lambda = .5$ against $H_2: \lambda = .8$

$s = .63829$ $D = .64122$

TABLE II - 1

$\alpha = .1, \; n_o = 87 \quad n_F^* = 47$

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1.	1.	11.	8.		0
.5	.9127	.90	31.14	27.38		13.22
.6	.6448		36.09			15.40
s	.5064	.50	36.07	34.24	31.56	15.80
D	.4958		36.03		30.85	15.83
.7	.2991		33.84			15.99
.8	.0923	.10	27.29	23.42		14.53

TABLE II - 2

$\alpha = .05, \; n_o = 126 \quad n_F^* = 78$

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1.	1.	15.	10.		0
.5	.9559	.95	46.87	40.88		18.41
.6	.6773		58.53			22.51
s	.5042	.50	58.93	61.49	53.15	23.23
D	.4908		58.85		51.82	23.28
.7	.2491		54.14			23.64
.8	.0458	.05	40.66	34.96		20.32

TABLE II - 3

$\alpha = .01, \quad n_o = 217 \quad n_F^* = 155$

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1.	1.	25.	16.		0
.5	.9911	.99	82.02	69.29		27.89
.6	.7368		116.5			38.15
s	.5018	.50	119.3	149.75	110.4	39.50
D	.4831		119.1		107.3	39.62
.7	.1693		104.8			40.82
.8	.0090	.01	70.18	59.26		30.58

TABLE II - 4

$\alpha = .001, \quad n_o = 350 \quad n_F^* = 274$

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1.	1.	40.	24.		0
.5	.9991	.999	130.4	106.1		36.81
.6	.7973		205.5			58.43
s	.5007	.50	215.3	338.3	201.9	60.08
D	.4760		214.9		195.8	60.36
.7	.1025		179.3			63.20
.8	.00089	.001	110.5	90.69		40.10

A.3.3. TABLES III - 1. 2. 3. 4.

H_1: λ = .5 against H_2: λ = 1.

s = .72135 D = .72850

TABLE III - 1

α = .1, n_o = 34 n_F = 22

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1.	1.	6.0	5.		0
.5	.9185	.90	13.81	11.60		5.40
.65	.6683		15.70			6.45
s	.5083	.50	15.62	13.93	12.87	6.74
D	.4929		15.58		12.41	6.76
.85	.2553		14.11			6.86
1.	.0877	.10	11.47	9.21		6.22

TABLE III - 2

α = .05, n_o = 52 n_F = 32

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1.	1.	9.	6.		0
.5	.9594	.95	20.33	17.32		7.49
.65	.7054		24.89			9.29
s	.5084	.50	25.02	25.02	21.67	9.76
D	.4884		24.95		20.84	9.80
.85	.2026		21.95			10.00
1.	.0442	.05	16.77	13.76		8.66

TABLE III - 3

$\alpha = .01, \quad n_o = 88 \quad n_F^* = 64$

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1.	1.	15.	10.		0
.5	.9915	.99	35.42	29.35		11.19
.65	.7655		48.69			15.51
s	.5014	.50	49.85	60.93	45.01	16.48
D	.4736		49.66		43.11	16.59
.85	.1185		40.82			17.04
1.	.0084	.01	28.22	23.32		12.81

TABLE III - 4

$\alpha = .001, \quad n_o = 142 \quad n_F^* = 113$

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1.	1.	23.	14.		0
.5	.9991	.999	56.07	44.93		14.71
.65	.8289		84.77			23.53
s	.5006	.50	89.25	137.64	82.34	24.92
D	.4640		88.87		78.61	25.17
.85	.0594		67.96			25.76
1.	.00082	.001	44.07	35.69		16.71

A.3.4. TABLES IV - 1. 2. 3. 4

H_1: λ = 5. against H_2: λ = 8.

s = 6.3829 D = 6.4122

TABLE IV - 1

α = .1, n_o = 9, n_F^* = 5

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1.	1.	2.	1.		0
5.	.9305	.90	3.80	2.74		1.59
6.	.6551		4.54			1.88
s	.5290	.50	4.80	3.42	3.16	1.88
D	.4919		4.57		3.08	1.91
7.	.2793		4.31			1.91
8.	.0728	.10	3.48	2.34		1.69

TABLE IV - 2

α = .05, n_o = 14 , n_F^* = 8

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1.	1.	2.	1.		0
5.	.9649	.95	5.39	4.09		2.04
6.	.6900		6.87			2.52
s	.5427	.50	7.43	6.15	5.31	2.52
D	.4908		6.91		5.18	2.61
7.	.2374		6.37			2.64
8.	.0367	.05	4.78	3.50		2.22

TABLE IV - 3

$\alpha = .01$, $n_o = 22$, $n_F^* = 16$

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1.	1.	3.	2.		0
5.	.9929	.99	8.92	6.93		2.94
6.	.7447		12.82			4.03
s	.5070	.50	13.28	14.98	11.04	4.09
D	.4806		13.11		10.73	4.19
7.	.1605		11.56			4.31=
8.	.0070	.01	7.76	5.93		3.18

TABLE IV - 4

$\alpha = .001$, $n_o = 35$, $n_F^* = 28$

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1.	1.	4.	3.		0
5.	.9993	.999	13.74	10.60		3.77
6.	.8010		21.70			6.05
s	.4989	.50	22.80	33.83	20.19	6.18
D	.4735		22.75		19.58	6.21
7.	.0969		19.00			6.50
8.	.00069	.001	11.81	9.07		4.09

APPENDIX IV

A.4. TABLES V - VI: Characteristics of the Poisson MPRT (λ_o = D and s) and SPRT Approximation

A.4.1. TABLES V - 1. 2. 3. 4

H_1: λ = .1 against H_2: λ = .5

s = .24853 D = .26129

TABLE V - 1

α = .1, n_o = 18, (n_{os} = 18), n_F = 11

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1. (1.)	1.	7. (7.)	6.		0. (0.)
.1	.9378 (.9392)	.90	9.08 (8.92)	7.44		2.80 (2.56)
.2	.6856 (.6933)		9.75 (9.51)			3.66 (3.43)
s	.5391 (.5496)	.50	9.50 (9.28)	7.50	7.03	3.92 (3.71)
D	.5022 (.5131)		9.39 (9.18)		6.37	3.98 (3.76)
.4	.1988 (.2094)		7.70 (7.59)			4.09 (3.95)
.5	.0909 (.0980)	.10	6.45 (6.40)	4.40		3.78 (3.69)

TABLE V - 2

$\alpha = .05, \quad n_o = 26 \quad (n_{os} = 26) \quad n_F = 19$

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1. (1.)	1.	10. (10.)	8.		0. (0.)
.1	.9646 (.9663)	.95	13.43 (12.84)	11.12		3.72 (3.53)
.2	.6975 (.7059)		15.08 (14.60)			5.21 (5.07)
s	.5167 (.5278)	.50	14.64 (14.30)	13.47	11.85	5.73 (5.51)
D	.4711 (.4826)		14.43 (14.13)		10.68	5.84 (5.60)
.4	.1302 (.1397)		11.00 (11.04)			5.95 (5.65)
.5	.0417 (.0472)	.05	8.72 (8.89)	6.57		5.20 (4.95)

TABLE V - 3

$\alpha = .01, \; n_o = 47 \; (n_{os} = 47) \; n_F = 38$

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1. (1.)	1.	17. (16.)	12.		0. (0.)
.1	.9924 (.9935)	.99	23.06 (22.16)	18.84		5.25 (5.43)
.2	.7473 (.7606)		28.73 (28.41)			8.34 (8.45)
s	.5059 (.5218)	.50	28.21 (28.30)	32.80	24.69	9.52 (9.42)
D	.4435 (.4591)		27.70 (27.88)		22.07	9.80 (9.67)
.4	.0567 (.0616)		18.94 (19.49)			9.58 (9.57)
.5	.0080 (.0092)	.01	14.05 (14.54)	11.13		7.48 (7.56)

TABLE V - 4

α = .001, n_o = 78 (n_{os} = 78) n_F = 63

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1. (1.)	1.	27. (25.)	18.		0. (0.)
.1	.9993 (.9993)	.999	36.42 (34.96)	28.83		7.05 (7.11)
.2	.8084 (.8151)		50.13 (49.02)			12.45 (12.45)
s	.5066 (.5175)	.50	50.26 (49.85)	74.10	45.24	14.58 (14.05)
D	.4247 (.4358)		49.21 (49.00)		40.22	15.20 (14.57)
.4	.0185 (.0205)		29.98 (30.84)			13.90 (13.60)
.5	.00076 (.00092)	.001	21.31 (22.14)	17.03		9.80 (9.76)

A.4.2. TABLES VI - 1. 2. 3. 4.

H_1: λ = 1. against H_2: λ = 2.

s = 1.4427 D = 1.4570

- -

TABLE VI - 1

$\alpha = .1$, $n_o = 17$, ($n_{os} = 11$), $n_F = 11$

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1. (1.)	1.	3. (3.)	3.		0. (0.)
1.	.9236 (.9299)	.90	7.33 (7.24)	5.80		2.86 (2.83)
1.35	.6150 (.6275)		8.44 (8.47)			3.46 (3.44)
s	.5095 (.5205)	.50	8.41 (8.45)	6.97	6.43	3.55 (3.51)
D	.4916 (.5040)		8.36 (8.44)		6.21	3.56 (3.52)
1.75	.2104 (.2193)		7.35 (7.52)			3.58 (3.53)
2.	.0817 (.0867)	.10	6.16 (6.36)	4.61		3.27 (3.24)

TABLE VI - 2

$\alpha = .05,\ n_o = 26,\ (n_{os} = 26),\ n_F^* = 16$

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1. (1.)	1.	5. (5.)	3.		0. (0.)
1.	.9620 (.9635)	.95	10.58 (10.46)	8.66		3.87 (3.86)
1.35	.6406 (.6462)		13.16 (13.13)			4.90 (4.93)
s	.5100 (.5140)	.50	13.17 (13.13)	12.51	10.83	5.04 (5.06)
D	.4879 (.4936)		13.09 (13.10)		10.42	5.07 (5.08)
1.75	.1573 (.1607)		11.07 (11.15)			5.12 (5.08)
2.	.0413 (.0427)	.05	8.80 (8.91)	6.88		4.47 (4.41)

TABLE VI - 3

$\alpha = .01,\ n_0 = 44,\ (n_{0s} = 44),\ n_F^* = 32$

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1. (1.)	1.	8. (8.)	5.		0. (0.)
1.	.9922 (.9926)	.99	18.13 (17.78)	14.68		5.67 (5.70)
1.35	.6824 (.6875)		25.57 (25.39)			8.05 (8.08)
s	.5066 (.5083)	.50	25.85 (25.60)	30.46	22.51	8.30 (8.34)
D	.4743 (.4802)		25.58 (25.52)		21.56	8.43 (8.39)
1.75	.0792 (.0820)		19.83 (20.00)			8.44 (8.37)
2.	.0079 (.0084)	.01	14.58 (14.76)	11.66		6.50 (6.50)

TABLE VI - 4

$\alpha = .001,\ n_o = 71,\ (n_{os} = 71),\ n_F^* = 57$

λ	OC		ASN		HLB	SDN
	MPRT	SPRT*	MPRT	SPRT*		MPRT
0	1. (1.)	1.	12. (12.)	7.		0. (0.)
1.	.9992 (.9993)	.999	28.44 (27.90)	22.46		7.40 (7.42)
1.35	.7315 (.7358)		44.73 (44.41)			12.07 (12.15)
s	.5016 (.5069)	.50	45.43 (45.33)	68.82	41.17	12.58 (12.52)
D	.4648 (.4701)		45.24 (45.18)		39.31	12.71 (12.63)
1.75	.0320 (.0333)		32.06 (32.40)			12.34 (12.24)
2.	.00076 (.00083)	.001	22.50 (22.84)	17.84		8.42 (8.39)

APPENDIX V

A.5. TABLE VII : Characteristics of the Poisson MPRT (λ_o = D), $SPRT_o$, SPRT approximation and FSST

TABLES VII - 1, 2, 3, 4

H_1: λ = .01 against H_2: λ = .1

s = .03909 D = .04299

A.5.1. TABLE VII - 1

α = .1, n_o = 58, n_F = 39, n_s = 22.20, n_D = 19.08

λ	OC			ASN		SDN	
	MPRT	$SPRT_o$	FSST	MPRT	$SPRT_o$	MPRT	$SPRT_o$
.0	1.	1.	1.	29.	25.	0.	0.
.0025	.9958	.9955	1.⁻	29.98	26.36	3.84	6.03
.006	.9776	.9820	.98	31.04	27.98	5.60	8.90
.01	.9432	.9566 .90*	.94	31.84	29.44 26.57*	6.86	10.89
.015	.8858	.9116	.88	32.36	30.72	8.03	12.53
.02	.8187	.8553	.81	32.43	31.45	8.98	13.68
.025	.7469	.7916	.75	32.16	31.73	9.80	14.52
.03	.6743	.7240	.67	31.64	31.63	10.50	15.17
.035	.6033	.6555	.60	30.94	31.24	11.08	15.66
s	.5479	.6005 .50*	.55	30.26	30.74 23.30*	11.48	15.56
D	.4976	.5495	.50	29.56	30.16	11.79	16.17

A.5.1. TABLE VII - 1 (continued)

$\alpha = .1,\ n_o = 58,\ n_F = 39,\ n_s = 22.20,\ n_D = 19.08$

λ	OC			ASN		SDN	
	MPRT	$SPRT_o$	FSST	MPRT	$SPRT_o$	MPRT	$SPRT_o$
.055	.3628	.4090	.37	27.21	27.93	12.37	16.43
.07	.2369	.2732	.25	24.50	24.81	12.44	16.06
.08	.1759	.2061	.18	22.41	22.78	12.22	15.52
.1	.0950	.1157 .10*	.10	19.18	19.17 12.69*	11.40	14.08
.11	.0694	.0865	.07	17.81	17.63	10.91	13.29

A.5.2. TABLE VII - 2

$\alpha = .05,\ n_o = 85,\ n_F = 63,\ n_s = 37.53,\ n_D = 32.03$

λ	OC			ASN		SDN	
	MPRT	$SPRT_o$	FSST	MPRT	$SPRT_o$	MPRT	$SPRT_o$
.0	1.	1.	1.	41.	33.	0.	0.
.0025	.9986	.9989	$1.^-$	42.52	35.18	5.00	7.86
.006	.9896	.9923	.99	44.50	38.23	7.94	12.73
.01	.9664	.9750 .95*	.97	46.39	41.42 39.68*	10.32	16.14
.015	.9175	.9363	.93	48.03	44.68	12.61	19.22
.02	.8501	.8792	.86	48.83	46.96	14.50	21.32

A.5.2. TABLE VII - 2 (continued)

$\alpha = .05,\ n_o = 85,\ n_F = 63,\ n_s = 37.53,\ n_D = 32.03$

λ	OC			ASN		SDN	
	MPRT	$SPRT_o$	FSST	MPRT	$SPRT_o$	MPRT	$SPRT_o$
.025	.7703	.8076	.79	48.85	48.22	16.13	22.83
.03	.6844	.7268	.71	48.23	48.57	17.54	23.95
.035	.5978	.6423	.62	47.08	48.15	18.74	24.81
s	.5292	.5738	.55	45.86	47.34	19.54	25.32
		.50*			41.84*		
D	.4672	.5106	.49	44.52	46.26	20.15	25.67
.55	.3054	.3418	.33	39.82	41.81	21.12	25.89
.07	.1681	.1944	.19	33.86	35.51	20.80	24.65
.08	.1097	.1307	.11	30.27	31.57	19.97	23.20
.1	.0448	.0581	.05	24.33	25.04	17.68	19.79
		.05*			18.95*		
.11	.0282	.0388	.03	21.96	22.46	16.45	18.09

TABLE VII - 3

$\alpha = .01,\ n_o = 143,\ n_F = 117,\ n_s = 78.39,\ n_D = 66.26$

λ	OC			ASN		SDN	
	MPRT	$SPRT_o$	FSST	MPRT	$SPRT_o$	MPRT	SPRT
.0	1.	1.	1.	67.	52.	0.	0.
.0025	.9999	.9999	$1.^-$	71.69	55.47	6.59	9.97
.006	.9991	.9991	$1.^-$	75.75	61.06	10.96	17.30
.01	.9931	.9938	.99	80.62	68.19	14.95	24.19

A.5.3. TABLE VII - 3 (continued)

$\alpha = .01,\ n_o = 143,\ n_F = 117,\ n_s = 78.39,\ n_D = 66.26$

λ	OC			ASN		SDN	
	MPRT	$SPRT_o$	FSST	MPRT	$SPRT_o$	MPRT	$SPRT_o$
.01		.99*			67.25*		
.015	.9681	.9717	.97	86.39	77.28	18.90	30.68
.02	.9146	.9223	.91	90.92	85.29	22.02	34.97
.025	.8315	.8428	.83	93.55	91.18	24.78	37.73
.03	.7264	.7397	.72	94.05	94.43	27.50	39.75
.035	.6107	.6245	.61	92.59	95.02	30.11	41.49
s	.5164	.5296	.52	90.19	93.77	32.00	42.78
		.50*			101.89*		
D	.4311	.4435	.44	87.14	91.39	33.48	43.82
.055	.2233	.2327	.23	75.39	79.66	35.47	44.99
.07	.0832	.0900	.09	60.69	62.90	33.36	41.61
.08	.0401	.0456	.05	52.52	53.27	30.55	37.56
.1	.0084	.0115	.01	40.46	39.21	24.66	28.99
		.01*			32.11*		
.11	.0037	.0059	.05	36.10	34.31	22.15	25.31

A.5.4.

TABLE VII - 4

$\alpha = .001$, $n_o = 243$, $n_F = 212$, $n_S = 143.97$, $n_D = 120.84$

λ	OC			ASN		SDN	
	MPRT	$SPRT_o$	FSST	MPRT	$SPRT_o$	MPRT	$SPRT_o$
.0	1.	1.	1.	110.	77.	0	0
.0025	.9999	.9999	1^-	114.0	82.33	7.92	12.50
.006	.9999	.9999	1^-	120.3	91.07	13.55	22.34
.01	.9993	.9994	1^-	128.5	103.3	19.41	33.76
		.999*			102.9*		
.015	.9917	.9937	.99	140.0	121.9	26.13	47.39
.02	.9610	.9685	.97	151.3	142.1	31.53	57.45
.025	.8903	.9050	.91	160.2	160.2	35.75	62.96
.03	.7763	.7958	.81	164.8	172.7	39.95	65.63
.035	.6331	.6529	.67	164.0	177.8	44.78	67.97
S	.5102	.5272	.55	159.9	176.7	48.78	69.61
		.50*			230.2*		
D	.3992	.4124	.44	153.5	171.1	52.05	73.07
.055	.1526	.1564	.18	127.3	140.9	55.50	76.19
.07	.0327	.0338	.04	96.64	100.8	48.71	65.68
.08	.0101	.0110	.01	81.41	81.01	42.26	55.22
.1	.00077	.0012	.001	61.05	56.20	31.68	37.96
		.001*			49.15*		
.11	.00020	.00044	.00025	54.16	48.51	27.91	32.07

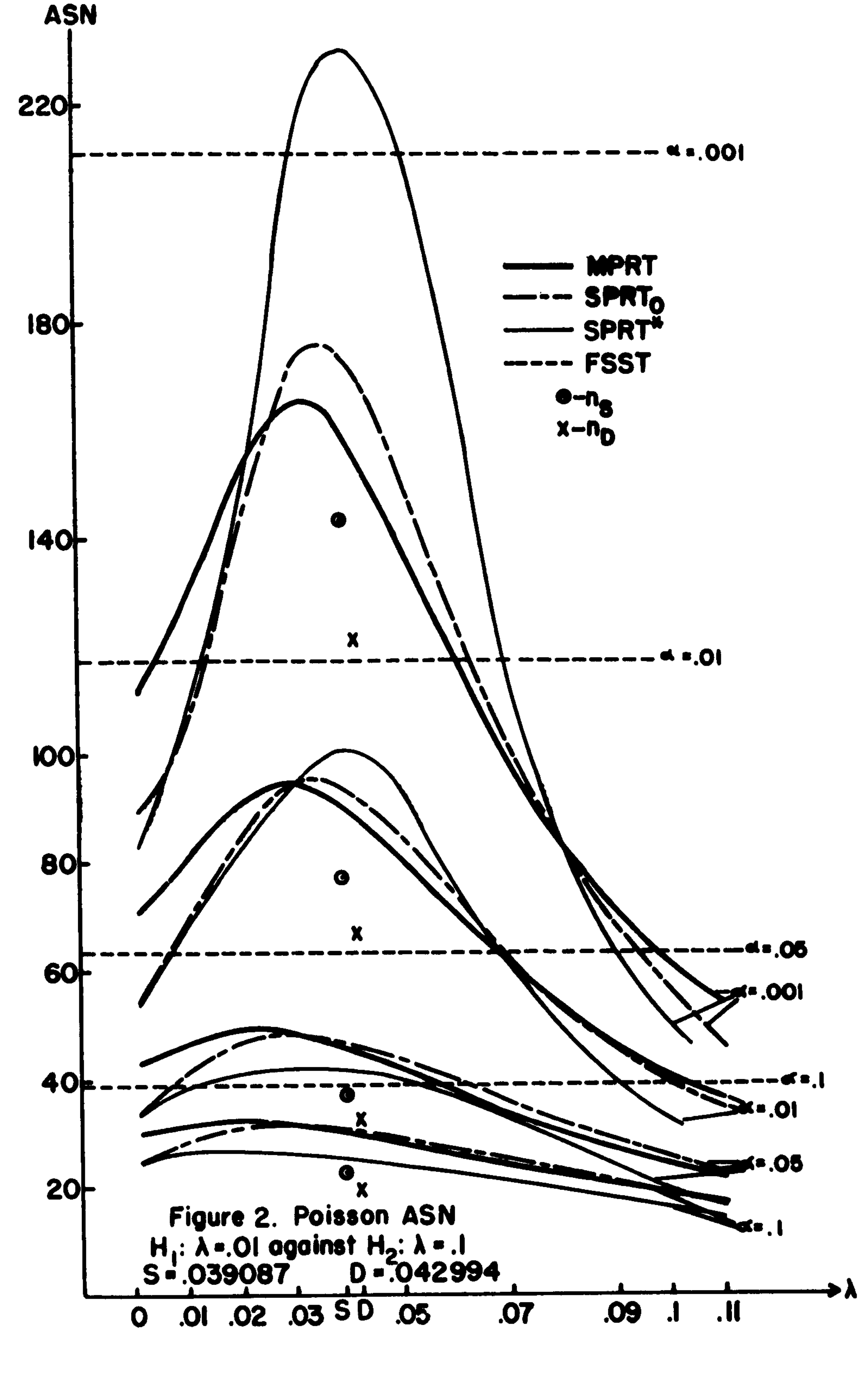

Figure 2. Poisson ASN
H_1: λ=.01 against H_2: λ=.1
S=.039087 D=.042994

APPENDIX VI

A.6. TABLE VIII: Characteristics of the Poisson MPRT (λ_o = D), $SPRT_o$, SPRT approximation and FSST

TABLES VIII - 1, 2, 3, 4

H_1: λ = 2. against H_2: λ = 4.

s = 2.8854 D = 2.9140

- -

A.6.1. TABLE VIII - 1

α = .1, n_o = 10, n_F = 6, n_s = 3.22, n_D = 3.10

λ	OC			ASN		SDN	
	MPRT	$SPRT_o$	FSST	MPRT	$SPRT_o$	MPRT	$SPRT_o$
.0	1.	1.	1.	2.	2.	0.	0.
1.5	.9925	.9932	.99	3.27	3.08	1.05	1.34
1.75	.9738	.9738	.98	3.62	3.56	1.27	1.76
2.0	.9298	.9410 .90*	.94	4.00	4.12 2.90*	1.48	2.19
2.25	.8491	.8658	.86	4.33	4.68	1.64	2.54
2.5	.7307	.7462	.75	4.56	5.11	1.75	2.78
2.65	.6463	.6572	.67	4.63	5.27	1.80	2.87
2.75	.5871	.5938	.61	4.64	5.32	1.83	2.91
s	.5062	.5065 .50*	.54	4.62	5.32 3.48*	1.86	2.95
D	.4893	.4883	.52	4.61	5.31	1.86	2.95
3.25	.3070	.2938	.34	4.37	4.98	1.89	2.93

A.6.1. TABLE VIII - 1 (continued)

$\alpha = .1,\ n_o = 10,\ n_F = 6,\ n_s = 3.22,\ n_D = 3.10$

λ	OC			ASN		SDN	
	MPRT	$SPRT_o$	FSST	MPRT	$SPRT_o$	MPRT	$SPRT_o$
3.5	.2016	.1858	.23	4.08	4.56	1.86	2.80
3.75	.1254	.1116	.14	3.77	4.10	1.79	2.60
4.	.0746	.0649	.09	3.45	3.64	1.69	2.34
		.1*			2.30*		
4.5	.0240	.0211	.03	2.90	2.90	1.45	1.82

A.6.2. TABLE VIII - 2

$\alpha = .05,\ n_o = 13,\ n_F^* = 8,\ n_s = 5.42,\ n_D = 5.21$

λ	OC			ASN		SDN	
	MPRT	$SPRT_o$	FSST*	MPRT	$SPRT_o$	MPRT	$SPRT_o$
.0	1.	1.	1.	3.	2.	0.	0.
1.5	.9984	.9984	.99	4.35	3.76	1.38	1.65
1.75	.9912	.9925	.99	4.96	4.48	1.72	2.28
2.0	.9647	.9711	.95	5.65	5.42	2.04	2.95
		.95*			4.33*		
2.25	.8973	.9118	.86	6.33	6.45	2.29	3.50
2.5	.7740	.7934	.72	6.85	7.34	2.46	3.82
2.65	.6761	.6944	.62	7.02	7.68	2.53	3.91
2.75	.6047	.6206	.55	7.06	7.80	2.58	3.95
s	.5052	.5166	.46	7.03	7.83	2.63	3.99
		.50*			6.25*		

A.6.2. TABLE VIII - 2

$\alpha = .05,\ n_o = 13,\ n_F^* = 8,\ n_s = 5.42,\ n_D = 5.21$

λ	OC			ASN		SDN	
	MPRT	$SPRT_o$	FSST*	MPRT	$SPRT_o$	MPRT	$SPRT_o$
D	.4843	.4947	.44	7.01	7.82	2.65	3.99
3.25	.2648	.2644	.25	6.51	7.20	2.73	3.94
3.5	.1496	.1466	.15	5.93	6.42	2.68	3.75
3.75	.0772	.0752	.08	5.30	5.59	2.54	3.42
4.	.0370	.0369 .05*	.05	4.72	4.83 3.44*	2.33	3.00
4.5	.0073	.0088	.01	3.78	3.70	1.87	2.20

A.6.3. TABLE VIII - 3

$\alpha = .01,\ n_o = 22,\ n_F^* = 16,\ n_s = 11.25,\ n_D = 10.78$

λ	OC			ASN		SDN	
	MPRT	$SPRT_o$	FSST*	MPRT	$SPRT_o$	MPRT	$SPRT_o$
.0	1.	1.	1.	4.	3.	0.	0.
1.5	.9999	.9999	$1.^-$	7.05	5.61	1.67	2.11
1.75	.9993	.9993	$1.^-$	8.06	6.79	2.23	2.96
2.	.9928	.9941 .99*	.99	9.39	8.51 7.33*	2.91	4.20
2.25	.9567	.9641	.94	11.00	10.84	3.57	5.49
2.5	.8442	.8601	.79	12.54	13.31	3.97	6.27

A.6.3. TABLE VIII - 3 (continued)

$\alpha = .01,\ n_o = 22,\ n_F^* = 16,\ n_s = 11.25,\ n_D = 10.78$

λ	OC			ASN		SDN	
	MPRT	$SPRT_o$	FSST*	MPRT	$SPRT_o$	MPRT	$SPRT_o$
2.65	.7285	.7458	.66	13.16	14.41	4.10	6.43
2.75	.6356	.6511	.57	13.37	14.84	4.17	6.49
s	.5008	.5112 .50*	.55	13.35	14.98 15.23*	4.30	6.56
D	.4723	.4814	.42	13.30	14.95	4.32	6.58
3.25	.1908	.1880	.17	11.88	13.11	4.52	6.65
3.5	.0757	.0723	.07	10.32	10.93	4.31	6.18
3.75	.0250	.0237	.03	8.84	8.89	3.84	5.30
4.	.0071	.0072 .01*	.003	7.62	7.29 5.83*	3.29	4.32
4.5	.00042	.00075	.001	5.93	5.27	2.39	2.84

A.6.4. TABLE VIII - 4

$\alpha = .001,\ n_o = 36,\ n_F^* = 28,\ n_s = 20.59,\ n_D = 19.65$

λ	OC			ASN		SDN	
	MPRT	$SPRT_o$	FSST*	MPRT	$SPRT_o$	MPRT	$SPRT_o$
.0	1.	1.	1.	6.	4.	0	0
1.5	.9999	.9999	1^-	10.83	7.86	2.10	2.49
1.75	.9999	.9999	1^-	12.42	9.58	2.79	3.65
2.	.9993	.9994 .999*	1^-	14.55	12.25 11.23*	3.75	5.53

A.6.4. TABLE VIII - 4 (continued)

$\alpha = .001,\ n_o = 36,\ n_F^* = 28,\ n_s = 20.59,\ n_D = 19.65$

λ	OC			ASN		SDN	
	MPRT	$SPRT_o$	FSST*	MPRT	$SPRT_o$	MPRT	$SPRT_o$
2.25	.9871	.9898	.99	17.45	16.51	4.97	8.13
2.5	.9060	.9151	.91	20.85	22.15	5.89	10.01
2.65	.7856	.7943	.79	22.51	25.15	6.08	10.22
2.75	.6738	.6783	.68	23.16	26.46	6.16	10.15
s	.5005	.4963	.47	23.26	26.99	6.38	10.18
		.50*			34.41*		
D	.4634	.4574	.44	23.16	26.91	6.44	10.21
3.25	.1269	.1143	.14	19.79	22.17	6.88	10.64
3.5	.0307	.0255	.04	16.43	17.10	6.24	9.41
3.75	.0053	.0044	.01	13.64	13.09	5.19	7.37
4.	.00069	.00074	.001	11.57	10.38	4.24	5.57
		.001*			8.92*		
4.5	.00071	.00034	.00023	8.89	7.32	2.99	3.46

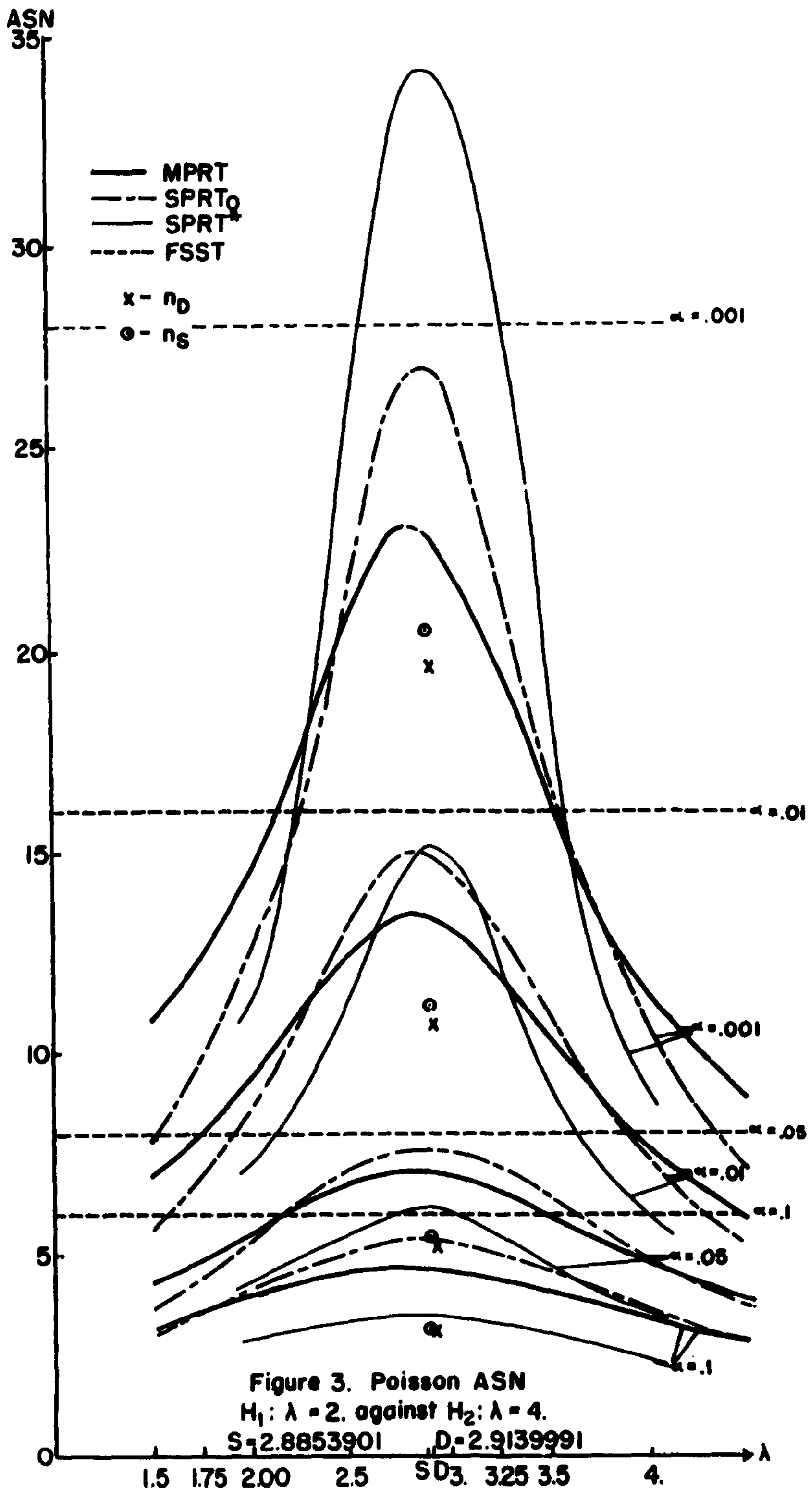

Figure 3. Poisson ASN
$H_1: \lambda = 2.$ against $H_2: \lambda = 4.$
S = 2.8853901 D = 2.9139991

COMPILER PROGRAM

A.7. K. Fukushima, MPRT-A

```
N 1200  Y 2400  Z 2400  S 0150  W 0200                    N
n905=$these $        f
n906=$are va$        f
n907=$lues o$        f
n908=$f c on$        f
n909=$e, d o$        f
n910=$ne, c $        f
n911=$two an$        f
n912=$d d tw$        f
n913=$o     $        f
n914=$this i$        f
n915=$s the $        f
n916=$upper $        f
n917=$value $        f
n918=$for po$        f
n919=$isson $        f
n920=$this i$        f
n921=$s the $        f
n922=$end of$        f
n923=$trial $        f
n924=$these $        f
n925=$are th$        f
n926=$e poss$        f
n927=$ible v$        f
n928=$alues $        f
n929=$of max$        f
n930=$ defec$        f
n931=$tives $        f
n932=$these $        f
n933=$two va$        f
n934=$lues m$        f
n935=$ust be$        f
n936=$equal. $       f
n937=$these $        f
n938=$are th$        f
n939=$e uppe$        f
n940=$r and $        f
n941=$lower $        f
n942=$bounda$        f
n943=$ries  $        f
n944=$these $        f
n945=$are th$        f
n946=$e acce$        f
n947=$ptance$        f
n948=$ proba$        f
n949=$biliti$        f
n950=$es for$        f
```

```
n951=$ each $     f
n952=$trial $     f
n953=$these $     f
n954=$are th$     f
n955=$e reje$     f
n956=$ction $     f
n957=$probab$     f
n958=$ilitie$     f
n959=$s for $     f
n960=$each t$     f
n961=$rial  $     f
n962=$this i$     f
n963=$s the $     f
n964=$prob. $     f
n965=$of acc$     f
n966=$eptanc$     f
n967=$e     $     f
n968=$this i$     f
n969=$s the $     f
n970=$prob. $     f
n971=$of rej$     f
n972=$ection$     f
n973=$this i$     f
n974=$s the $     f
n975=$averag$     f
n976=$e numb$     f
n977=$er of $     f
n978=$sample$     f
n979=$s     $     f
n980=$this i$     f
n981=$s the $     f
n982=$varian$     f
n983=$ce of $     f
n984=$number$     f
n985=$ of sa$     f
n986=$mples $     f
n987=$this i$     f
n988=$s the $     f
n989=$end of$     f
n990=$ a par$     f
n991=$t of c$     f
n992=$alcula$     f
n993=$tion  $     f
n994=$these $     f
n995=$are po$     f
n996=$isson $     f
n997=$prob. $     f
n998=$these $     f
n999=$are th$     f
n1000=$e cumu$    f
n1001=$lative$    f
n1002=$ poiss$    f
```

```
        n1003=$on pro$        f
        n1004=$b. fro$        f
        n1005=$m zero$        f
        n1006=$these $        f
        n1007=$are th$        f
        n1008=$e cumu$        f
        n1009=$lative$        f
        n1010=$ poiss$        f
        n1011=$on pro$        f
        n1012=$b. up $        f
        n1013=$to inf$        f
        n1014=$inity $        f
        n1015=$this i$        f
        n1016=$s the $        f
        n1017=$end of$        f
        n1018=$ trial$        f
        n1019=$ pract$        f
        n1020=$ically$        f
        n1021=$this i$        f
        n1022=$s the $        f
        n1023=$s.d. o$        f
        n1024=$f numb$        f
        n1025=$er of $        f
        n1026=$sample$        f
        n1027=$s     $        f
0001    y1,...,y4   y28,...,y38 fukushima input
0116    ty1 ty2 ty3 ty4       f
0130    ty28 ty29 ty30        f
0117    ty31 ty32 ty33 ty34             f
0118    ty35 ty36 ty37 ty38             f
        z1=(*01s,y3* - *01s,y1*)        f
        z2=(*01s,y3* - *01s,y2*)        f
        z3=(*01s,y2* - *01s,y1*)        f
        z4=(*01s,2xy4*)       f
        y5=z4/z2  f
        y6=(y3-y2)/z2         f
        y7=((-1)xz4)/z3       f
        y8=(y2-y1)/z3         f
0080    ty5 ty6 ty7 ty8       f
0079    atn905 atn913         f
        n902=y7-y5+5          f
        tn902     f
        atn914 atn919         f
        y9=(y7-y5)/(y6-y8)    f
0078    ty9       f
0077    atn920 atn923         f
        y10=y5+y6xy9          f
        y11=y7+y8xy9          f
0076    ty10 ty11             f
0075    atn924 atn931         f
0074    atn932 atn936         f
        n1=0      f
0002    y(1000+n1)=y5+y6xn1             f
```

```
       zn1=y7+y8xn1          f
       g3 if y(1000+n1) v 0          f
       y(1000+n1)=(-1.)      f
0003   n(10+n1)=y(1000+n1)           f
       n(500+n1)=zn1         f
       g4 if zn1-n(500+n1) u 0       f
       n(500+n1)=n(500+n1)+1         f
0004   y0=y0        f
       tn(500+n1) tn1 tn(10+n1)      f
       n1=n1+1      f
       g2 if n(500+n1-1)-n(10+n1-1) v 1      f
       n900=n1-1             f
       atn937 atn943         f
0072   tn900        f
0071   atn1015 atn1020       f      The end of Part A.
       y39=y2       f
       n901=0       f
       y2=y28       f
       g40 if y2 u 0         f
       g5           f
0040   n901=1       f
       y2=y29       f
       g41 if y2 u 0         f
       g5           f
0041   n901=2       f
       y2=y30       f
       g120 if y2 u 0        f
       g5           f
0120   n901=3       f
       y2=y31       f
       g121 if y2 u 0        f
       g5           f
0121   n901=4       f
       y2=y32       f
       g122 if y2 u 0        f
       g5           f
0122   n901=5       f
       y2=y33       f
       g123 if y2 u 0        f
       g5           f
0123   n901=6       f
       y2=y34       f
       g124 if y2 u 0        f
       g5           f
0124   n901=7       f
       y2=y35       f
       g125 if y2 u 0        f
       g5           f
0125   n901=8       f
       y2=y36       f
       g126 if y2 u 0        f
       g5           f
```

```
0126  n901=9     f
      y2=y37     f
      g127 if y2 u 0       f
      g5         f
0127  n901=10    f
      y2=y38     f
      g42 if y2 u 0        f
      g5         f
0005  y1000=(*02s,(-1)xy2*)          f
      ty1000     f
      7,n2,1,1,n902,       f
0006  y12=n2     f
      y(1000+n2)=(y(1000+n2-1)xy2)/y12          f
      n2=y12     f
      ty(1000+n2) tn2      f
0007  y0=y0      f
      atn994 atn997        f      The end of Part B.
      z1000=y1000          f
      tz1000 t(0.)         f
      9,n2,1,1,n902,       f
0008  z(1000+n2)=z(1000+n2-1)+y(1000+n2)        f
      tz(1000+n2) tn2      f
0009  y0=y0      f
      atn998 atn1005       f      The end of Part C.
      z0=1       f
      t(1.) t(0.)          f
      51,n2,1,1,n902,      f
0050  zn2=1-z(1000+n2-1)   f
      tzn2 tn2   f
0051  y0=y0      f
      atn1006 atn1014      f      The end of Part D.
      n1=1       f
      n1050=n501-1         f
0011  12,n2,n11,1,n1050,   f
0012  y(100+n2)=y(1000+n2)           f
      g13 if n11 u (-1)    f
      y2001=y(100+n11)     f
      ty2001 t(1.)         f
      g14        f
0013  y2001=0    f
      ty2001 t(1.)         f
0014  z2001=zn501          f
      t(0.) t(1.) tz2001   f      The end of Part E.
0015  n1=n1+1    f                Part F
0017  g21 if n(10+n1) w 0           f   Part G

         y(2000+n1)=0         f
         t(0.) tn1            f
         n1051=n(500+n1)-1    f
         n1094=n(500+n1-1)    f
         n1095=n(500+n1-1)-1          f
         20,n2,0,1,n1095,     f
```

```
0018  y(500+n2)=0          f
0019  16,n3,0,1,n2,        f
0016  y(500+n2)=y(500+n2)+y(100+n3)xy(1000+n2-n3)                          f
      ty(500+n2) tn1 tn2   f
0020  y0=y0      f
      g28 if n(500+n1-1) u n(500+n1)          f
      94,n2,n1094,1,n1051,            f
0091  y(500+n2)=0          f
0092  93,n3,0,1,n1095,     f
0093  y(500+n2)=y(500+n2)+y(100+n3)xy(1000+n2-n3)                          f
      ty(500+n2)  tn1 tn2             f
0094  y0=y0      f
      g28        f           The end of Part H.
0021  g22 if n(10+n1)-n(10+n1-1) v 0          f
      y(2000+n1)=0         f
      t(0.) tn1            f
      n1054=n(10+n1)+1     f
      n1055=n(500+n1)-1    f
      g25        f
0022  y(2000+n1)=0         f
      n1052=n(10+n1-1)+1   f
      n1053=n(10+n1)       f
      n1054=n(10+n1)+1     f
      n1055=n(500+n1)-1    f
      23,n4,n1052,1,n1053,            f
0023  y(2000+n1)=y(2000+n1)+y(100+n4)xz(1000+n(10+n1)-n4)                  f
      ty(2000+n1) tn1      f
      atn944 atn952        f
0025  n1051=n(500+n1)-1    f
      n1094=n(500+n1-1)    f
      n1095=n(500+n1-1)-1             f
      n1052=n(10+n1-1)+1   f
      98,n2,n1054,1,n1095,            f
0026  y(500+n2)=0          f
      27,n3,n1052,1,n2,    f
0027  y(500+n2)=y(500+n2)+y(100+n3)xy(1000+n2-n3)                          f
      ty(500+n2)  tn1055  tn2  tn3  f
0098  y0=y0      f
      g28 if n(500+n1-1) u  n(500+n1)         f
      90,n2,n1094,1,n1051,            f
0085  y(500+n2)=0          f
0086  87,n3,n1052,1,n1095,            f
0087  y(500+n2)=y(500+n2)+y(100+n3)xy(1000+n2-n3)                          f
      ty(500+n2)  tn1 tn2             f
0090  y0=y0      f           The end of Part I.
0028  n2=n(500+n1)         f
      n1057=n(10+n1-1)+1   f
      n1058=n(500+n1-1)-1             f
      y15=0      f
      29,n5,n1057,1,n1058,            f
0029  y15=y15+y(100+n5)xz(n2-n5)      f
      z(2000+n1)=y15       f
```

```
     tn1 tz(2000+n1)       f
     atn953 atn961         f      The end of Part J.
     n1061=n(10+n1)+1      f
     n1062=n(500+n1)-1     f
     31,n2,n1061,1,n1062,           f
0031 y(100+n2)=y(500+n2)            f
     g32 if n1 w n900      f      Part K
     g15      f
0032 y20=0    f
     y21=0    f
     y22=0    f
     y23=0    f
     34,n1,1,1,n900,       f
0033 y20=y20+y(2000+n1)    f
     y21=y21+z(2000+n1)    f
     y22=y22+n1x(y(2000+n1)+z(2000+n1))        f
0034 y23=y23+n1xn1x(y(2000+n1)+z(2000+n1))     f
0105 ty20 tn900 ty2 ty4    f
0106 atn962 atn967         f
0107 ty21 tn900 ty2 ty4    f
0108 atn968 atn972         f
0109 ty22 ty2 ty4          f
0110 atn973 atn979         f
     y24=y23-(y22xy22)     f
0111 ty24 ty2 ty4          f
0112 atn980 atn986         f
     y25=(*06s,y24*)       f
0113 ty25     f
0114 atn1021 atn1027       f
0115 atn987 atn993         f      The end of Part L.
     g40 if n901 u 0       f
     g41 if n901 u 1       f
     g120 if n901 u 2      f
     g121 if n901 u 3      f
     g122 if n901 u 4      f
     g123 if n901 u 5      f
     g124 if n901 u 6      f
     g125 if n901 u 7      f
     g126 if n901 u 8      f
     g127 if n901 u 9      f
     g42 if n901 u 10      f      The end of Par M.
0042 y2=y39   f
0043 g1       ff
```

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